

MTAEA – Vectors in Euclidean Spaces

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Vectors.

- ▶ Economists usually work in the vector space \mathbb{R}^n . A point in this space is called a **vector**, and is typically defined by its rectangular coordinates.
- ▶ For instance, let $v \in \mathbb{R}^n$. We define this vector by its n coordinates, v_1, v_2, \dots, v_n . It is common to write $v = (v_1, v_2, \dots, v_n)$ or to display a vector as a column matrix:

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

- ▶ It is common to distinguish between locations and displacements by writing a location as a row vector and a displacement as a column vector. However, we can use the same algebraic operations to work with each.

Vectors.

- ▶ A vector can be also be defined by its origin and end points.
- ▶ Suppose the vector v links the point $P = (p_1, \dots, p_n)$ to the point $Q = (q_1, \dots, q_n)$ in \mathbb{R}^n . Then $v = Q - P$, i.e. $v_i = p_i - q_i$, $\forall i \in \{1, 2, \dots, n\}$.

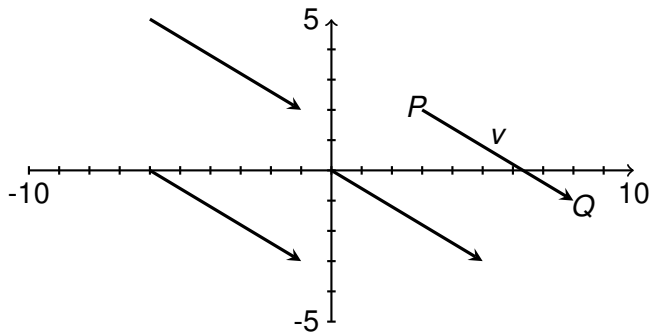


Figure: The displacement $(5, -3)$

Addition.

- ▶ Given two vectors u and v , with coordinates, we add them like so:

$$u + v = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix}.$$

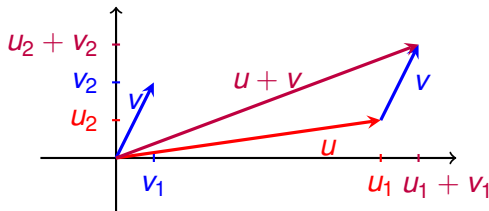


Figure: Vector Addition

Scalar Multiplication.

- ▶ We can also multiply vectors by scalars. Suppose $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^n$. Scalar multiplication gives a vector $\lambda v \in \mathbb{R}^n$, defined by

$$\lambda v = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{pmatrix}$$

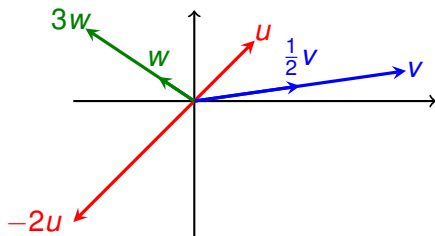


Figure: Scalar multiplication

Subtraction.

- ▶ The difference of two vectors, say $u - v$, is the sum of the vector u with the vector $-v = (-1)v$.

$$u - v = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + (-1) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \\ \vdots \\ u_n - v_n \end{pmatrix}.$$

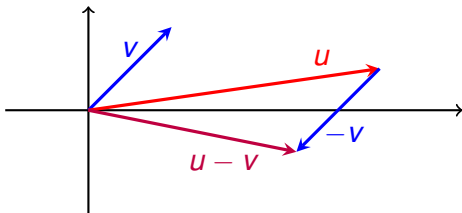


Figure: Vector Subtraction

Laws of Vector Algebra.

- ▶ Let $\lambda, \beta \in \mathbb{R}$ and $u, v, w \in \mathbb{R}^n$. Then the following algebraic properties of vectors hold.
 - ▶ Associativity:

$$\begin{aligned}(u + v) + w &= u + (v + w) \\ (\lambda\beta)v &= \lambda(\beta v)\end{aligned}$$

- ▶ Commutativity:

$$u + v = v + u$$

- ▶ Distributivity:

$$\begin{aligned}\lambda(u + v) &= \lambda u + \lambda v \\ (\lambda + \beta)u &= \lambda u + \beta u\end{aligned}$$

Definition.

Two vectors $u, v \in \mathbb{R}^n$ are **parallel** if there exists a real number λ such that $u = \lambda v$.



The Inner Product.

Definition.

Let u and v be two vectors in \mathbb{R}^n . The **(Euclidean) inner product** (also called **dot product** or **scalar product**) of u and v is the real number, denoted $u \cdot v$, given by

$$\begin{aligned} u \cdot v &= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \\ &= \sum_{i=1}^n u_i v_i \end{aligned}$$



- ▶ Note that if we think of u and v as $n \times 1$ matrices, we have $u \cdot v = u^T v = v^T u$.
- ▶ If we think of them as $1 \times n$ matrices, then $u \cdot v = uv^T = vu^T$.

Laws of the Inner Product.

Let $\lambda \in \mathbb{R}$ and $u, v, w \in \mathbb{R}^n$. Then:

- ▶ Associativity:

$$(\lambda u) \cdot v = \lambda(u \cdot v)$$

- ▶ Commutativity:

$$u \cdot v = v \cdot u$$

- ▶ Distributivity:

$$u \cdot (v + w) = u \cdot v + u \cdot w$$

Length and Inner Product.

Definition.

The **norm** or **length** of a vector u is the real number, denoted $\|u\|$, given by

$$\|u\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}. \quad \blacktriangle$$

- ▶ Using our definition of the inner product we can also write this as

$$\|u\| = \sqrt{u \cdot u}$$

- ▶ The norm of a vector is always positive unless the vector is the zero vector, in which case the norm is zero.
- ▶ The distance between two vectors $u, v \in \mathbb{R}^n$ is calculated as

$$\|u - v\| = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}.$$

- ▶ Note that for any $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}^n$:

$$\|\lambda u\| = |\lambda| \|u\|.$$

Length and Inner Product.

- ▶ We can show that

$$u \cdot v = \|u\| \|v\| \cos \theta$$

where θ is the angle between the vectors u and v .

- ▶ Using the properties of the cosine we get the following result.

Theorem

The angle between vectors u and v in \mathbb{R}^n is

- (i) acute, if $u \cdot v > 0$,
- (ii) obtuse, if $u \cdot v < 0$,
- (iii) right, if $u \cdot v = 0$.

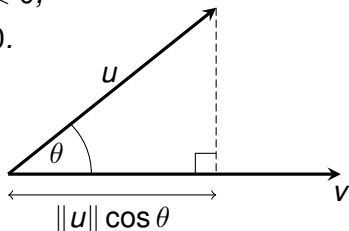


Figure: The angle between two vectors u and v .

Length and Inner Product.

Definition.

Let v be a vector. The vector w which points in the same direction as v , but has length 1 is called the **unit vector** in the direction of v (or simply the **direction** of v). It is given by

$$w = \frac{v}{\|v\|}. \quad \blacktriangle$$

Definition.

Two vectors $u, v \in \mathbb{R}^n$ are **orthogonal** if $u \cdot v = 0$. ▲

- ▶ This definition implies the zero vector is orthogonal to any vector.

Definition.

Two vectors $u, v \in \mathbb{R}^n$ are **orthonormal** if they are orthogonal and are unit vectors. ▲

Length and Inner Product.

Theorem (Triangle Inequality)

For any two vectors $u, v \in \mathbb{R}^n$,

$$\|u + v\| \leq \|u\| + \|v\|.$$

Theorem (Triangle Inequality Variant)

For any two vectors $u, v \in \mathbb{R}^n$,

$$|\|u\| - \|v\|| \leq \|u - v\|.$$

- ▶ There are three basic properties of Euclidean length for any vectors u and v and scalar λ :

1. $\|u\| \geq 0$ and $\|u\| = 0$ only when $u = 0$,
2. $\|\lambda u\| = |\lambda| \|u\|$,
3. $\|u + v\| \leq \|u\| + \|v\|$.

Any assignment of a real number to a vector satisfying these properties is called a norm (see sections 29.4 and 27 of S&B if interested).

Projections.

- ▶ Let $u, v \in \mathbb{R}^n$. We want to find the **vector projection** of the vector u in the direction of v .
- ▶ Denote the projection of u on v by $P_v(u)$. We can see from the diagram that the length of $P_v(u)$ (called the **scalar projection** of vector u on v) is given by

$$\|P_v(u)\| = \|u\| \cos \theta.$$

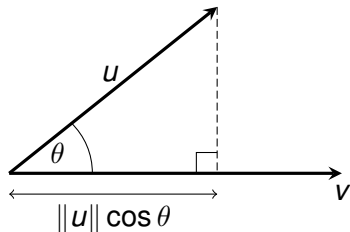


Figure: Projection of vector u in direction v

Projections.

- ▶ The unit vector in the direction of v is

$$\frac{1}{\|v\|} v.$$

So we have

$$P_v(u) = \frac{\|u\| \cos \theta}{\|v\|} v.$$

Using our expression for the inner product $u \cdot v = \|u\| \|v\| \cos \theta$ we get

$$P_v(u) = \frac{u \cdot v}{\|v\|^2} v.$$

- ▶ Note that if the vectors are parallel, say $u = \lambda v$ for some scalar λ , then $P_v(u) = u$. This says that projecting a vector u in its own direction gives the same vector u .

Lines.

- ▶ A straight line is completely determined by two things:
 - ▶ a point $p \in \mathbb{R}^n$ on the line, and
 - ▶ a direction $v \in \mathbb{R}^n$, with $v \neq 0$, in which to move from p .
- ▶ A straight line ℓ in \mathbb{R}^n can be defined in its parametric form as

$$\ell = \{x \in \mathbb{R}^n \mid x = p + tv, t \in \mathbb{R}\}$$

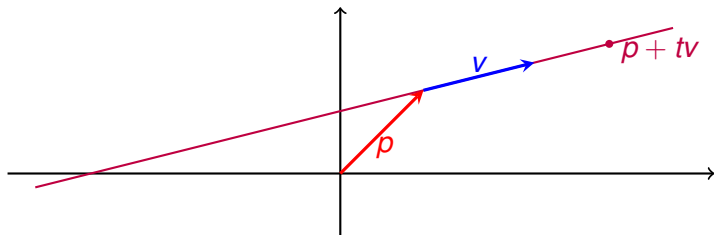


Figure: Parametric line ℓ in \mathbb{R}^2

Lines.

- ▶ So, a point $x \in \mathbb{R}^n$ belongs to the line ℓ iff there exists a real number t such that $x = p + tv$.
- ▶ A straight line ℓ can also be defined by two distinct points P and Q .
 - ▶ The vector $v = Q - P$ gives the direction of the line.
 - ▶ So then the line is just the set of points $P + tv$, with $t \in \mathbb{R}$.
- ▶ Now, $P + tv = P + t(Q - P) = (1 - t)P + tQ$, so the line is

$$\ell = \{(1 - t)P + tQ \in \mathbb{R}^n \mid t \in \mathbb{R}\}$$

Two-dimensional Planes.

- ▶ We saw that a line – a one-dimensional object – can be described using only one parameter.
- ▶ A plane is two-dimensional, so we need two parameters.
- ▶ Consider a plane \mathcal{P} in \mathbb{R}^3 and let u and v be two vectors in \mathcal{P} that point in different directions:

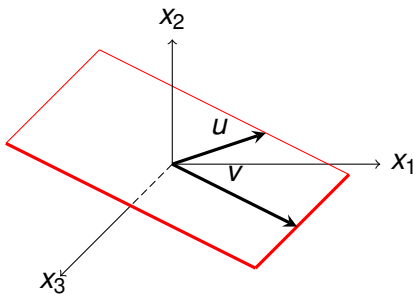


Figure: A plane \mathcal{P} through the origin in \mathbb{R}^3

Two-dimensional Planes.

- ▶ We can move from the origin in direction u , v or any combination of the two. So, for any scalars s and t , the vector $su + tv$ also lies in the plane \mathcal{P} .
- ▶ Thus any plane \mathcal{P} through the origin, in a vector space \mathbb{R}^n ($n > 2$), can be defined in its parametric form as

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid x = su + tv, s, t \in \mathbb{R}\}$$

- ▶ But what if the plane does not pass through the origin?

Two-dimensional Planes.

- ▶ Suppose the plane does not pass through the origin.

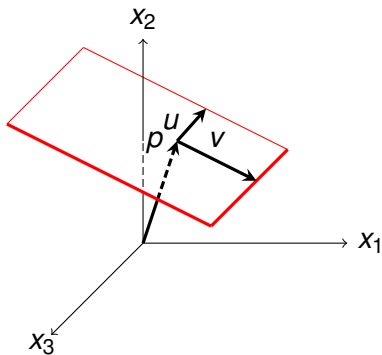


Figure: A plane \mathcal{P} not through the origin in \mathbb{R}^3

Two-dimensional Planes.

- ▶ We can move from point p in the plane in direction u , v or any combination of the two. Thus the vector $p + su + tv$ also lies in the plane P .
- ▶ So any plane \mathcal{P} through the point p , in a vector space \mathbb{R}^n ($n > 2$), can be defined in its parametric form as

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid x = p + su + tv, s, t \in \mathbb{R}\}$$

- ▶ A point $x \in \mathbb{R}^n$ belongs to the plane \mathcal{P} iff there exist two scalars s and t such that $x = p + su + tv$. Equivalently, the vector $x - p$ must be a linear combination of the vectors u and v .

Two-dimensional Planes.

- ▶ As two points uniquely determine a line, three distinct (non-collinear) points P , Q and R uniquely determine a plane.
 - ▶ Let $u = Q - P$ and $v = R - P$. We can picture these as displacement vectors from P :

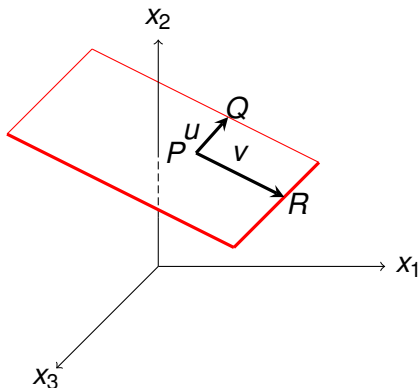


Figure: A plane \mathcal{P} not through the origin in \mathbb{R}^3

Two-dimensional Planes.

- ▶ Remember, we need u and v to be nonparallel to for them to uniquely determine a plane.
 - ▶ If u and v are parallel, then there exists a scalar λ such that $v = \lambda u$. But by definition of u and v , it would then be the case that $R = (1 - \lambda)P + \lambda Q$.
 - ▶ But then the points lie on a line ℓ and so are collinear.
- ▶ The plane containing the three points P , Q and R is

$$\mathcal{P} = \{(1 - s - t)P + sQ + tR \mid s, t \in \mathbb{R}\},$$

or equivalently

$$\mathcal{P} = \{t_P P + t_Q Q + t_R R \mid t_P, t_Q, t_R \in \mathbb{R} \text{ and } t_P + t_Q + t_R = 1\},$$

Two-dimensional Planes.

- ▶ We can also completely describe a plane using
 - ▶ a point $p = (x_0, y_0, z_0) \in \mathbb{R}^3$ in the plane, and
 - ▶ an inclination, specified by a vector $n = (a, b, c) \in \mathbb{R}^3$, called a **normal vector**, which is perpendicular to the plane.

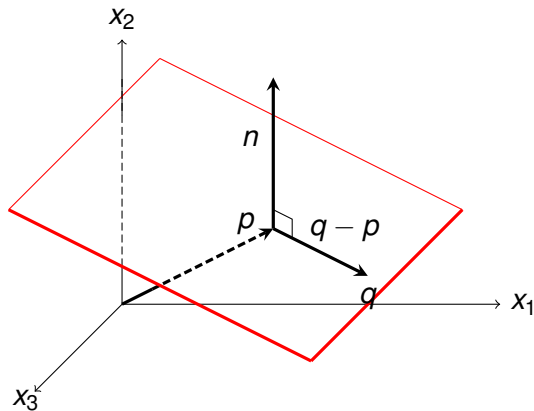


Figure: A plane \mathcal{P} through p with normal n

Two-dimensional Planes.

- ▶ Let $q = (x, y, z)$ be an arbitrary point on the plane \mathcal{P} . Then $q - p$ is a vector in the plane and will thus be perpendicular to n .
- ▶ Two vectors are orthogonal iff their dot product is zero, so the plane is defined as

$$\mathcal{P} = \{q \in \mathbb{R}^3 \mid n \cdot (q - p) = 0\}.$$

- ▶ Now, for any point q in the plane

$$0 = n \cdot (q - p) = (a, b, c) \cdot (x - x_0, y - y_0, z - z_0),$$

or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

- ▶ So another way to write the nonparametric equation of a plane, is

$$ax + by + cz = d,$$

where $d = ax_0 + by_0 + cz_0$. This is called the **point-normal equation** of the plane.

Hyperplanes.

- ▶ We saw a line in \mathbb{R}^2 can be written as

$$a_1 x_1 + a_2 x_2 = d$$

and a plane in \mathbb{R}^3 can be written in point-normal form as

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = d.$$

- ▶ Generalizing, a **hyperplane** in \mathbb{R}^n can be written in point-normal form as

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = d,$$

where (a_1, a_2, \dots, a_n) is a normal.

- ▶ The set of vectors in the hyperplane have tail at $(0, \dots, 0, d/a_n)$ and are perpendicular to the normal vector to the hyperplane.

Hyperplanes.

Example

- (1) An economic application you have probably seen deals with commodity spaces.

- ▶ The vector

$$x = (x_1, x_2, \dots, x_n)$$

of nonnegative quantities of n commodities is called a **commodity bundle**. The set of all commodity bundles is the set

$$\{(x_1, \dots, x_n) \mid x_1 \geq 0, \dots, x_n \geq 0\}$$

and is called a **commodity space**.

- ▶ Let $p_i > 0$ be the price of commodity i . The cost of buying bundle x is

$$p_1x_1 + p_2x_2 + \dots + p_nx_n = p \cdot x.$$

A consumer with income I can purchase only bundles x for which $p \cdot x \leq I$. This subset of the commodity space is the consumer's **budget set**.

Hyperplanes.

- ▶ The budget set is bounded above by the hyperplane $p \cdot x = I$, whose normal vector is the price vector p .

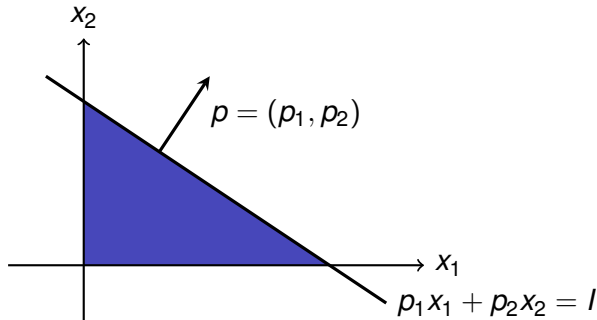


Figure: A consumer's budget set, $p \cdot x \leq I$, in commodity space.

Hyperplanes.

- (2) ▶ Another hyperplane you will see is the space of **probability vectors**

$$P_n = \{(p_1, \dots, p_n) \mid p_1 \geq 0, \dots, p_n \geq 0 \text{ and } p_1 + \dots + p_n = 1\},$$

which is called the **probability simplex**. The probability simplex P_n is part of a hyperplane in \mathbb{R}^n with normal vector $(1, 1, \dots, 1)$. ♦

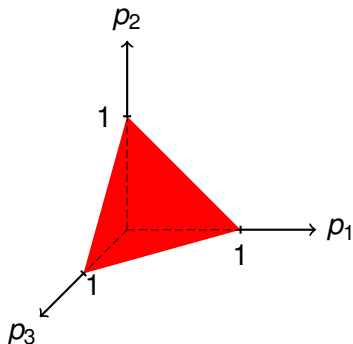


Figure: The probability simplex for $n = 3$.