# Vectors in Euclidean Spaces

**Vectors.**

- Economists usually work in the vector space $\mathbb{R}^n$. A point in this space is called a *vector*, and is typically defined by its rectangular coordinates.
- For instance, let $v \in \mathbb{R}^n$. We define this vector by its $n$ coordinates, $v_1, v_2, \ldots, v_n$. It is common to write $v = (v_1, v_2, \ldots, v_n)$ or to display a vector as a column matrix:

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

- It is common to distinguish between locations and dispacements by writing a location as a row vector and a displacement as a column vector. However, we can use the same algebraic operations to work with each.
- A vector can be also be defined by its origin and end points.
- Suppose the vector $v$ links the point $P = (p_1, \ldots, p_n)$ to the point $Q = (q_1, \ldots, q_n)$ in $\mathbb{R}^n$. Then $v = Q - P$, i.e. $v_i = p_i - q_i, \forall i \in \{1, 2, \ldots, n\}$.

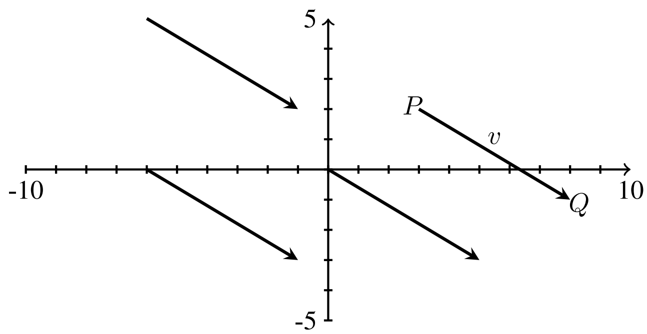


Figure 1: The displacement $(5, -3)$

**Addition.**

- Given two vectors $u$ and $v$, with coordinates, we add them like so:

$$u + v = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix}.$$

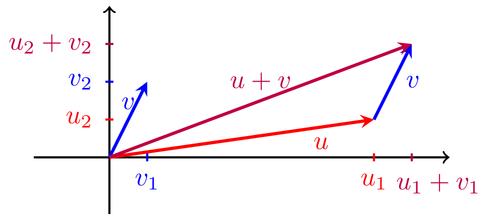


Figure 2: Vector Addition

**Scalar Multiplication.**

- We can also multiply vectors by scalars. Suppose $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^n$. Scalar multiplication gives a vector $\lambda v \in \mathbb{R}^n$, defined by

$$\lambda v = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{pmatrix}$$

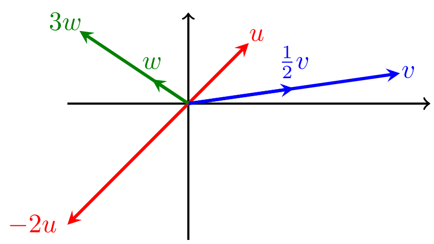


Figure 3: Scalar multiplication

**Subtraction.**

- The difference of two vectors, say $u - v$, is the sum of the vector $u$ with the vector $-v = (-1)v$.

$$u - v = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + (-1) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \\ \vdots \\ u_n - v_n \end{pmatrix}.$$

**Laws of Vector Algebra.**

- Let $\lambda, \beta \in \mathbb{R}$ and $u, v, w \in \mathbb{R}^n$. Then the following algebraic properties of vectors hold.

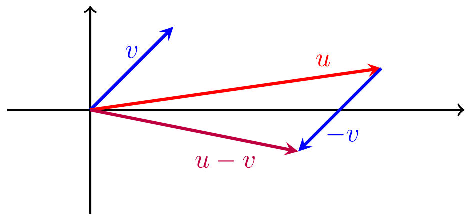


Figure 4: Vector Subtraction

– Associativity:

$$\begin{aligned}(u+v)+w &= u+(v+w)\\ (\lambda\beta)v &= \lambda(\beta v)\end{aligned}$$

– Commutativity:

$$u+v = v+u$$

– Distributivity:

$$\begin{aligned}\lambda(u+v) &= \lambda u+\lambda v\\ (\lambda+\beta)u &= \lambda u+\beta u\end{aligned}$$

***Definition.*** Two vectors $u, v \in \mathbb{R}^n$ are *parallel* if there exists a real number $\lambda$ such that $u = \lambda v$. ▲

**The Inner Product.**

***Definition.*** Let $u$ and $v$ be two vectors in $\mathbb{R}^n$. The *(Euclidean) inner product* (also called *dot product* or *scalar product*) of $u$ and $v$ is the real number, denoted $u \cdot v$, given by

$$\begin{aligned}u\cdot v &= u_1v_1+u_2v_2+\cdots+u_nv_n\\ &= \sum_{i=1}^{n} u_iv_i\end{aligned}$$ ▲

- Note that if we think of $u$ and $v$ as $n\times 1$ matrices, we have $u\cdot v = u^Tv = v^Tu$.
- If we think of them as $1\times n$ matrices, then $u\cdot v = uv^T = vu^T$.

**Laws of the Inner Product.**

Let $\lambda \in \mathbb{R}$ and $u, v, w \in \mathbb{R}^n$. Then:

- Associativity:

$$(\lambda u)\cdot v = \lambda(u\cdot v)$$

- Commutativity:

$$u \cdot v \quad = \quad v \cdot u$$

- Distributivity:

$$u \cdot (v + w) \quad = \quad u \cdot v + u \cdot w$$

**Length and Inner Product.**

***Definition.*** The *norm* or *length* of a vector $u$ is the real number, denoted $\|u\|$, given by

$$\|u\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}. \qquad \blacktriangle$$

- Using our definition of the inner product we can also write this as

$$\|u\| = \sqrt{u.u}$$

- The norm of a vector is always positive unless the vector is the zero vector, in which case the norm is zero.
- The distance between two vectors $u, v \in \mathbb{R}^n$ is calculated as

$$\|u - v\| = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}.$$

- Note that for any $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}^n$:

$$\|\lambda u\| = |\lambda|\|u\|.$$

- We can show that

$$u \cdot v = \|u\|\|v\| \cos\theta$$

  where $\theta$ is the angle between the vectors $u$ and $v$.
- Using the properties of the cosine we get the following result.

**Theorem 1.** *The angle between vectors $u$ and $v$ in $\mathbb{R}^n$ is*

1. *acute, if $u \cdot v > 0$,*
2. *obtuse, if $u \cdot v < 0$,*
3. *right, if $u \cdot v = 0$.*

***Definition.*** Let $v$ be a vector. The vector $w$ which points in the same direction as $v$, but has length 1 is called the *unit vector* in the direction of $v$ (or simply the *direction* of $v$). It is given by

$$w = \frac{v}{\|v\|}. \qquad \blacktriangle$$

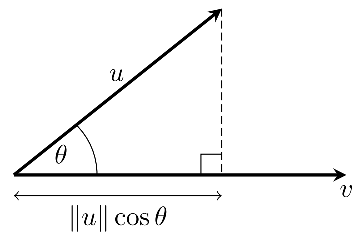

Figure 5: The angle between two vectors $u$ and $v$.

***Definition.*** Two vectors $u, v \in \mathbb{R}^n$ are *orthogonal* if $u \cdot v = 0$. ▲

- This definition implies the zero vector is orthogonal to any vector.

***Definition.*** Two vectors $u, v \in \mathbb{R}$ are *orthonormal* if they are orthogonal and are unit vectors. ▲

**Theorem 2** (Triangle Inequality)**.** *For any two vectors* $u, v \in \mathbb{R}^n$*,*

$$\|u + v\| \leq \|u\| + \|v\|.$$

**Theorem 3** (Triangle Inequality Variant)**.** *For any two vectors* $u, v \in \mathbb{R}^n$*,*

$$|\|u\| - \|v\|| \leq \|u - v\|.$$

- There are three basic properties of Euclidean length for any vectors $u$ and $v$ and scalar $\lambda$:

  1. $\|u\| \geq 0$ and $\|u\| = 0$ only when $u = 0$,
  2. $\|\lambda u\| = |\lambda|\|u\|$,
  3. $\|u + v\| \leq \|u\| + \|v\|$.

  Any assignment of a real number to a vector satisfying these properties is called a norm (see sections 29.4 and 27 of S&B if interested).

**Projections.**

- Let $u, v \in \mathbb{R}^n$. We want to find the *vector projection* of the vector $u$ in the direction of $v$.

- Denote the projection of $u$ on $v$ by $P_v(u)$. We can see from the diagram that the length of $P_v(u)$ (called the *scalar projection* of vector $u$ on $v$) is given by

$$\|P_v(u)\| = \|u\| \cos\theta.$$

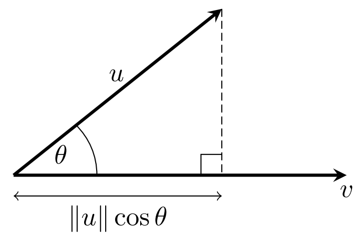


Figure 6: Projection of vector $u$ in direction $v$

- The unit vector in the direction of $v$ is
$$\frac{1}{\|v\|}v.$$
So we have
$$P_v(u) = \frac{\|u\|\cos\theta}{\|v\|}v.$$
Using our expression for the inner product $u \cdot v = \|u\|\|v\|\cos\theta$ we get
$$P_v(u) = \frac{u \cdot v}{\|v\|^2}v.$$

- Note that if the vectors are parallel, say $u = \lambda v$ for some scalar $\lambda$, then $P_v(u) = u$. This says that projecting a vector $u$ in its own direction gives the same vector $u$.

**Lines.**

- A straight line is completely determined by two things:
  - a point $p \in \mathbb{R}^n$ on the line, and
  - a direction $v \in \mathbb{R}^n$, with $v \neq 0$, in which to move from $p$.
- A straight line $\ell$ in $\mathbb{R}^n$ can be defined in its parametric form as
$$\ell = \{x \in \mathbb{R}^n \mid x = p + tv, t \in \mathbb{R}\}$$
- So, a point $x \in \mathbb{R}^n$ belongs to the line $\ell$ iff there exists a real number $t$ such that $x = p + tv$.
- A straight line $\ell$ can also be defined by two distinct points $P$ and $Q$.
  - The vector $v = Q - P$ gives the direction of the line.
  - So then the line is just the set of points $P + tv$, with $t \in \mathbb{R}$.
- Now, $P + tv = P + t(Q - P) = (1 - t)P + tQ$, so the line is
$$\ell = \{(1 - t)P + tQ \in \mathbb{R}^n \mid t \in \mathbb{R}\}$$

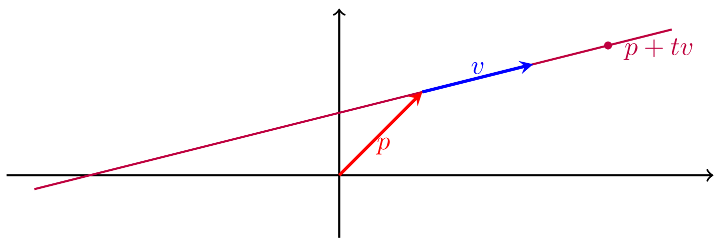


Figure 7: Parametric line $\ell$ in $\mathbb{R}^2$

**Two-dimensional Planes.**

- We saw that a line – a one-dimensional object – can be described using only one parameter.
- A plane is two-dimensional, so we need two parameters.
- Consider a plane $\mathcal{P}$ in $\mathbb{R}^3$ and let $u$ and $v$ be two vectors in $\mathcal{P}$ that point in different directions:

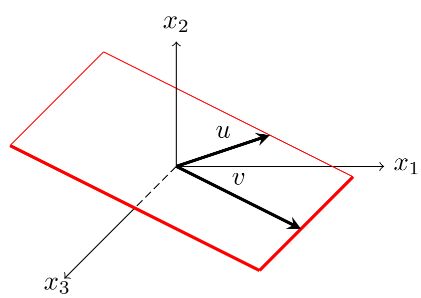


Figure 8: A plane $\mathcal{P}$ through the origin in $\mathbb{R}^3$

- We can move from the origin in direction $u$, $v$ or any combination of the two. So, for any scalars $s$ and $t$, the vector $su + tv$ also lies in the plane $\mathcal{P}$.
- Thus any plane $P$ through the origin, in a vector space $\mathbb{R}^n$ $(n > 2)$, can be defined in its parametric form as

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid x = su + tv, s, t \in \mathbb{R}\}$$

- But what if the plane does not pass through the origin?
- Suppose the plane does not pass throught the origin.
- We can move from point $p$ in the plane in direction $u$, $v$ or any combination of the two. Thus the vector $p + su + tv$ also lies in the plane $P$.

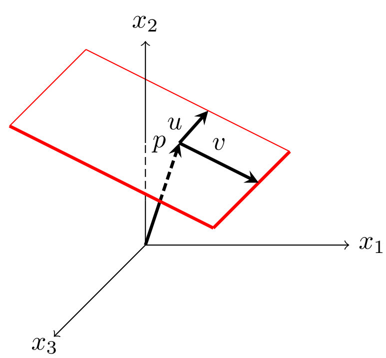


Figure 9: A plane $\mathcal{P}$ not through the origin in $\mathbb{R}^3$

- So any plane $\mathcal{P}$ through the point $p$, in a vector space $\mathbb{R}^n$ ($n > 2$), can be defined in its parametric form as

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid x = p + su + tv, s, t \in \mathbb{R}\}$$

- A point $x \in \mathbb{R}^n$ belongs to the plane $\mathcal{P}$ iff there exist two scalars $s$ and $t$ such that $x = p + su + tv$. Equivalently, the vector $x - p$ must be a linear combination of the vectors $u$ and $v$.

- As two points uniquely determine a line, three distinct (non-collinear) points $P$, $Q$ and $R$ uniquely determine a plane.
  - Let $u = Q - P$ and $v = R - P$. We can picture these as displacement vectors from $P$:

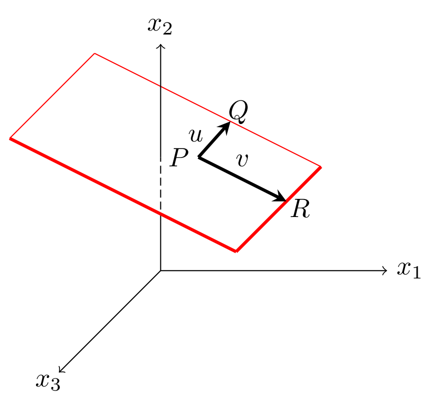


Figure 10: A plane $\mathcal{P}$ not through the origin in $\mathbb{R}^3$

- Remember, we need $u$ and $v$ to be nonparallel to for them to uniquely determine a plane.

  - If $u$ and $v$ *are* parallel, then there exists a scalar $\lambda$ such that $v = \lambda u$. But by definition of $u$ and $v$, it would then be the case that $R = (1 - \lambda)P + \lambda Q$.
  - But then the points lie on a line $\ell$ and so are collinear.

- The plane containing the three points $P$, $Q$ and $R$ is

$$\mathcal{P} = \{(1 - s - t)P + sQ + tR \mid s, t \in \mathbb{R}\},$$

  or equivalently

$$\mathcal{P} = \{t_P P + t_Q Q + t_R R \mid t_P, t_Q, t_R \in \mathbb{R} \text{ and } t_P + t_Q + t_R = 1\},$$

- We can also completely describe a plane using
  - a point $p = (x_0, y_0, z_0) \in \mathbb{R}^3$ in the plane, and
  - an inclination, specified by a vector $n = (a, b, c) \in \mathbb{R}^3$, called a *normal vector*, which is perpendicular to the plane.

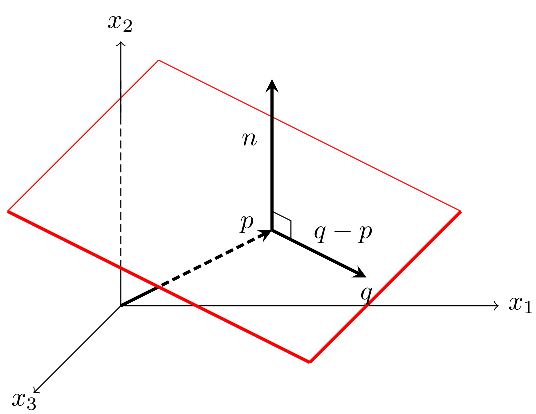


Figure 11: A plane $\mathcal{P}$ through $p$ with normal $n$

- Let $q = (x, y, z)$ be an arbitrary point on the plane $\mathcal{P}$. Then $q - p$ is a vector in the plane and will thus be perpendicular to $n$.

- Two vectors are orthogonal iff their dot product is zero, so the plane is defined as

$$\mathcal{P} = \{q \in \mathbb{R}^3 \mid n.(q - p) = 0\}.$$

- Now, for any point $q$ in the plane

$$0 = n.(q - p) = (a, b, c).(x - x_0, y - y_0, z - z_0),$$

  or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

- So another way to write the nonparametric equation of a plane, is

$$ax + by + cz = d,$$

  where $d = ax_0 + by_0 + cz_0$. This is called the *point-normal equation* of the plane.

**Hyperplanes.**

- We saw a line in $\mathbb{R}^2$ can be written as

$$a_1x_1 + a_2x_2 = d$$

and a plane in $\mathbb{R}^3$ can be written in point-normal form as

$$a_1x_1 + a_2x_2 + a_3x_3 = d.$$

- Generalizing, a *hyperplane* in $\mathbb{R}^n$ can be written in point-normal form as

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = d,$$

where $(a_1, a_2, \ldots, a_n)$ is a normal.

  - The set of vectors in the hyperplane have tail at $(0, \ldots, 0, d/a_n)$ and are perpindicular to the normal vector to the hyperplane.

**Example 1.**

1. An economic application you have probably seen deals with commodity spaces.

   - The vector

   $$x = (x_1, x_2, \ldots, x_n)$$

   of nonnegative quantities of $n$ commodities is called a *commodity bundle*. The set of all commodity bundles is the set

   $$\{(x_1, \ldots, x_n) \mid x_1 \geq 0, \ldots, x_n \geq 0\}$$

   and is called a *commodity space*.

   - Let $p_i > 0$ be the price of commodity $i$. The cost of buying bundle $x$ is

   $$p_1x_1 + p_2x_2 + \cdots + p_nx_n = p \cdot x.$$

   A consumer with income $I$ can purchase only bundles $x$ for which $p \cdot x \leq I$. This subset of the commodity space is the consumer's *budget set*.

1. - The budget set is bounded above by the hyperplane $p \cdot x = I$, whose normal vector is the price vector $p$.

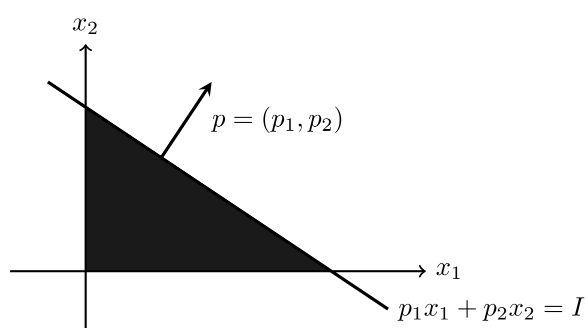

Figure 12: A consumer's budget set, $p \cdot x \leq I$, in commodity space.

2. • Another hyperplane you will see is the space of *probability vectors*

$$P_n = \{(p_1, \ldots, p_2) \mid p_1 \geq 0, \ldots, p_n \geq 0 \text{ and } p_1 + \cdots + p_n = 1\},$$

which is called the *probability simplex*. The probability simplex $P_n$ is part of a hyperplane in $\mathbb{R}^n$ with normal vector $(1, 1, \ldots, 1)$. ♦

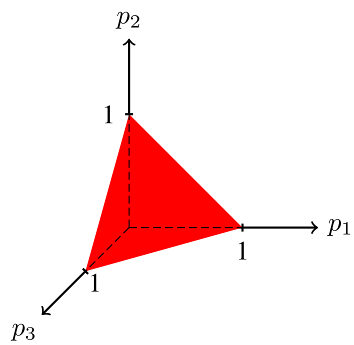


Figure 13: The probability simplex for $n = 3$.