Vectors in Euclidean Spaces

Vectors.

- Economists usually work in the vector space $\mathbb{R}^n$. A point in this space is called a *vector*, and is typically defined by its rectangular coordinates.

- For instance, let $v \in \mathbb{R}^n$. We define this vector by its n coordinates, $v_1, v_2, \ldots, v_n$. It is common to write $v = (v_1, v_2, \ldots, v_n)$ or to display a vector as a column matrix:

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

- It is common to distinguish between locations and dispacements by writing a location as a row vector and a displacement as a column vector. However, we can use the same algebraic operations to work with each.

- A vector can be also be defined by its origin and end points.

- Suppose the vector v links the point $P = (p_1, \ldots, p_n)$ to the point $Q = (q_1, \ldots, q_n)$ in $\mathbb{R}^n$. Then $v = Q - P$, i.e. $v_i = p_i - q_i, \forall i \in \{1, 2, \ldots, n\}$.

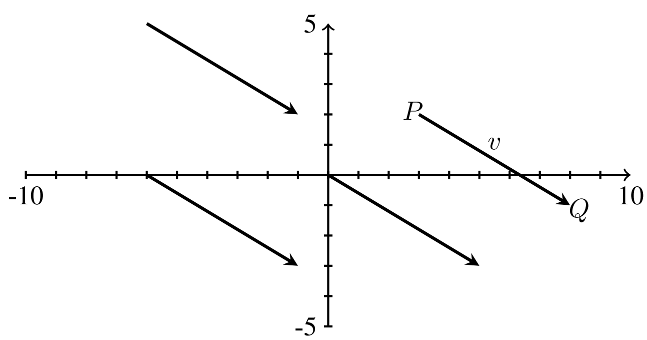

Figure 1: The displacement $(5, -3)$

Addition.

- Given two vectors u and v, with coordinates, we add them like so:

$$u + v = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{pmatrix}.$$

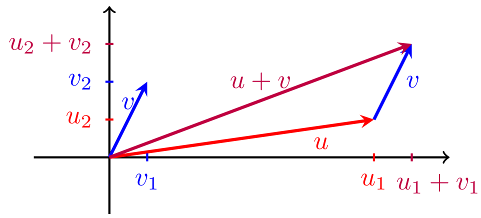

Figure 2: Vector Addition

Scalar Multiplication.

- We can also multiply vectors by scalars. Suppose $\lambda \in \mathbb{R}$ and $v \in \mathbb{R}^n$. Scalar multiplication gives a vector $\lambda v \in \mathbb{R}^n$, defined by

$$\lambda v = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{pmatrix}$$

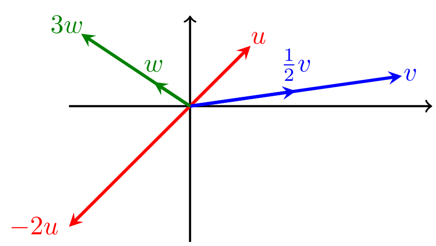

Figure 3: Scalar multiplication

Subtraction.

- The difference of two vectors, say $u - v$, is the sum of the vector u with the vector $-v = (-1)v$.

$$u - v = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} + (-1) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \\ \vdots \\ u_n - v_n \end{pmatrix}.$$

Laws of Vector Algebra.

- Let $\lambda, \beta \in \mathbb{R}$ and $u, v, w \in \mathbb{R}^n$. Then the following algebraic properties of vectors hold.

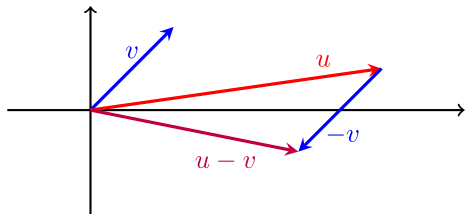

Figure 4: Vector Subtraction

- Associativity:

$$\begin{aligned}(u+v)+w &= u+(v+w) \\ (\lambda\beta)v &= \lambda(\beta v)\end{aligned}$$

- Commutativity:

$$u+v = v+u$$

- Distributivity:

$$\begin{aligned}\lambda(u+v) &= \lambda u+\lambda v \\ (\lambda+\beta)u &= \lambda u+\beta u\end{aligned}$$

Definition. Two vectors $u, v \in \mathbb{R}^n$ are *parallel* if there exists a real number λ such that $u = \lambda v$. ▲

The Inner Product.

Definition. Let u and v be two vectors in $\mathbb{R}^n$. The *(Euclidean) inner product* (also called *dot product* or *scalar product*) of u and v is the real number, denoted $u \cdot v$, given by

$$\begin{aligned}u \cdot v &= u_1v_1 + u_2v_2 + \cdots + u_nv_n \\ &= \sum_{i=1}^{n} u_iv_i\end{aligned}$$ ▲

- Note that if we think of u and v as $n \times 1$ matrices, we have $u \cdot v = u^Tv = v^Tu$.
- If we think of them as $1 \times n$ matrices, then $u \cdot v = uv^T = vu^T$.

Laws of the Inner Product.

Let $\lambda \in \mathbb{R}$ and $u, v, w \in \mathbb{R}^n$. Then:

- Associativity:

$$(\lambda u) \cdot v = \lambda(u \cdot v)$$

- Commutativity:

$$u \cdot v \quad = \quad v \cdot u$$

- Distributivity:

$$u \cdot (v + w) \quad = \quad u \cdot v + u \cdot w$$

Length and Inner Product.

Definition. The *norm* or *length* of a vector u is the real number, denoted $\|u\|$, given by

$$\|u\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}. \qquad \blacktriangle$$

- Using our definition of the inner product we can also write this as

$$\|u\| = \sqrt{u.u}$$

- The norm of a vector is always positive unless the vector is the zero vector, in which case the norm is zero.

- The distance between two vectors $u, v \in \mathbb{R}^n$ is calculated as

$$\|u - v\| = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2}.$$

- Note that for any $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}^n$:

$$\|\lambda u\| = |\lambda|\|u\|.$$

- We can show that

$$u \cdot v = \|u\|\|v\| \cos\theta$$

  where θ is the angle between the vectors u and v.

- Using the properties of the cosine we get the following result.

Theorem 1. *The angle between vectors u and v in $\mathbb{R}^n$ is*

1. *acute, if $u \cdot v > 0$,*
2. *obtuse, if $u \cdot v < 0$,*
3. *right, if $u \cdot v = 0$.*

Definition. Let v be a vector. The vector w which points in the same direction as v, but has length 1 is called the *unit vector* in the direction of v (or simply the *direction* of v). It is given by

$$w = \frac{v}{\|v\|}. \qquad \blacktriangle$$

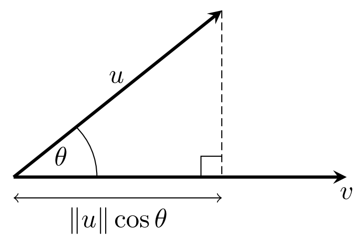

Figure 5: The angle between two vectors u and v.

Definition. Two vectors $u, v \in \mathbb{R}^n$ are *orthogonal* if $u \cdot v = 0$. ▲

- This definition implies the zero vector is orthogonal to any vector.

Definition. Two vectors $u, v \in \mathbb{R}$ are *orthonormal* if they are orthogonal and are unit vectors. ▲

Theorem 2 (Triangle Inequality)**.** *For any two vectors* $u, v \in \mathbb{R}^n$*,*

$$\|u + v\| \leq \|u\| + \|v\|.$$

Theorem 3 (Triangle Inequality Variant)**.** *For any two vectors* $u, v \in \mathbb{R}^n$*,*

$$|\|u\| - \|v\|| \leq \|u - v\|.$$

- There are three basic properties of Euclidean length for any vectors u and v and scalar λ:

    1. $\|u\| \geq 0$ and $\|u\| = 0$ only when $u = 0$,
    2. $\|\lambda u\| = |\lambda|\|u\|$,
    3. $\|u + v\| \leq \|u\| + \|v\|$.

    Any assignment of a real number to a vector satisfying these properties is called a norm (see sections 29.4 and 27 of S&B if interested).

Projections.

- Let $u, v \in \mathbb{R}^n$. We want to find the *vector projection* of the vector u in the direction of v.

- Denote the projection of u on v by $P_v(u)$. We can see from the diagram that the length of $P_v(u)$ (called the *scalar projection* of vector u on v) is given by

$$\|P_v(u)\| = \|u\| \cos\theta.$$

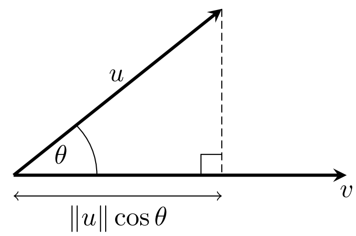

Figure 6: Projection of vector u in direction v

- The unit vector in the direction of v is
$$\frac{1}{\|v\|}v.$$
So we have
$$P_v(u) = \frac{\|u\|\cos\theta}{\|v\|}v.$$
Using our expression for the inner product $u \cdot v = \|u\|\|v\|\cos\theta$ we get
$$P_v(u) = \frac{u \cdot v}{\|v\|^2}v.$$
- Note that if the vectors are parallel, say $u = \lambda v$ for some scalar λ, then $P_v(u) = u$. This says that projecting a vector u in its own direction gives the same vector u.

Lines.

- A straight line is completely determined by two things:
  - a point $p \in \mathbb{R}^n$ on the line, and
  - a direction $v \in \mathbb{R}^n$, with $v \neq 0$, in which to move from p.
- A straight line ℓ in $\mathbb{R}^n$ can be defined in its parametric form as
$$\ell = \{x \in \mathbb{R}^n \mid x = p + tv, t \in \mathbb{R}\}$$
- So, a point $x \in \mathbb{R}^n$ belongs to the line ℓ iff there exists a real number t such that $x = p + tv$.
- A straight line ℓ can also be defined by two distinct points P and Q.
  - The vector $v = Q - P$ gives the direction of the line.
  - So then the line is just the set of points $P + tv$, with $t \in \mathbb{R}$.
- Now, $P + tv = P + t(Q - P) = (1 - t)P + tQ$, so the line is
$$\ell = \{(1 - t)P + tQ \in \mathbb{R}^n \mid t \in \mathbb{R}\}$$

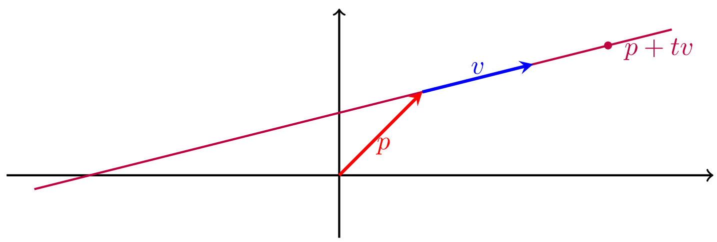

Figure 7: Parametric line ℓ in $\mathbb{R}^2$

Two-dimensional Planes.

- We saw that a line – a one-dimensional object – can be described using only one parameter.
- A plane is two-dimensional, so we need two parameters.
- Consider a plane $\mathcal{P}$ in $\mathbb{R}^3$ and let u and v be two vectors in $\mathcal{P}$ that point in different directions:

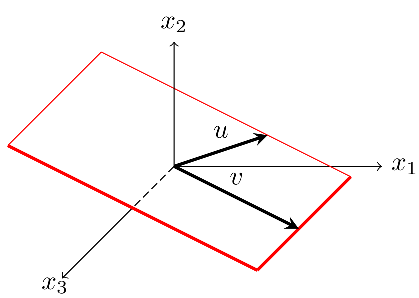

Figure 8: A plane $\mathcal{P}$ through the origin in $\mathbb{R}^3$

- We can move from the origin in direction u, v or any combination of the two. So, for any scalars s and t, the vector $su + tv$ also lies in the plane $\mathcal{P}$.
- Thus any plane P through the origin, in a vector space $\mathbb{R}^n$ ($n > 2$), can be defined in its parametric form as

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid x = su + tv, s, t \in \mathbb{R}\}$$

- But what if the plane does not pass through the origin?
- Suppose the plane does not passs throught the origin.
- We can move from point p in the plane in direction u, v or any combination of the two. Thus the vector $p + su + tv$ also lies in the plane P.

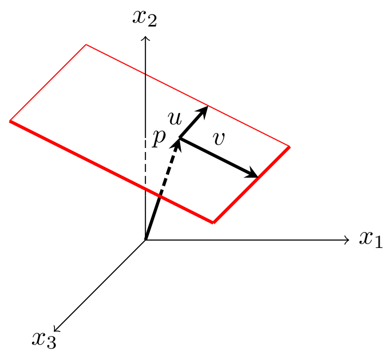

Figure 9: A plane $\mathcal{P}$ not through the origin in $\mathbb{R}^3$

- So any plane $\mathcal{P}$ through the point p, in a vector space $\mathbb{R}^n$ ($n > 2$), can be defined in its parametric form as

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid x = p + su + tv, s, t \in \mathbb{R}\}$$

- A point $x \in \mathbb{R}^n$ belongs to the plane $\mathcal{P}$ iff there exist two scalars s and t such that $x = p + su + tv$. Equivalently, the vector $x - p$ must be a linear combination of the vectors u and v.

- As two points uniquely determine a line, three distinct (non-collinear) points P, Q and R uniquely determine a plane.
  - Let $u = Q - P$ and $v = R - P$. We can picture these as displacement vectors from P:

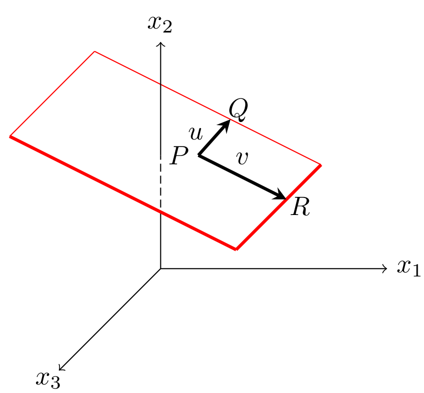

Figure 10: A plane $\mathcal{P}$ not through the origin in $\mathbb{R}^3$

- Remember, we need u and v to be nonparallel to for them to uniquely determine a plane.

- If u and v *are* parallel, then there exists a scalar λ such that $v = \lambda u$. But by definition of u and v, it would then be the case that $R = (1 - \lambda)P + \lambda Q$.
- But then the points lie on a line ℓ and so are collinear.

- The plane containing the three points P, Q and R is

$$\mathcal{P} = \{(1 - s - t)P + sQ + tR \mid s, t \in \mathbb{R}\},$$

or equivalently

$$\mathcal{P} = \{t_P P + t_Q Q + t_R R \mid t_P, t_Q, t_R \in \mathbb{R} \text{ and } t_P + t_Q + t_R = 1\},$$

- We can also completely describe a plane using
  - a point $p = (x_0, y_0, z_0) \in \mathbb{R}^3$ in the plane, and
  - an inclination, specified by a vector $n = (a, b, c) \in \mathbb{R}^3$, called a *normal vector*, which is perpendicular to the plane.

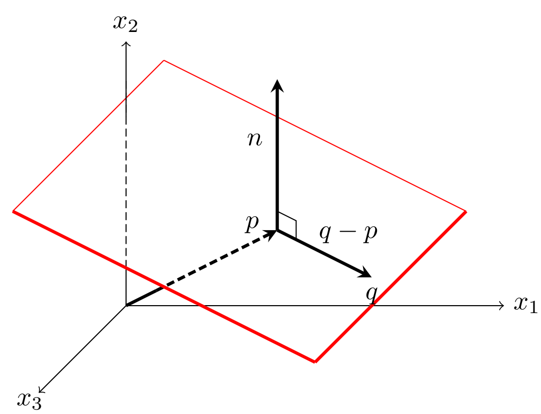

Figure 11: A plane $\mathcal{P}$ through p with normal n

- Let $q = (x, y, z)$ be an arbitrary point on the plane $\mathcal{P}$. Then $q - p$ is a vector in the plane and will thus be perpendicular to n.
- Two vectors are orthogonal iff their dot product is zero, so the plane is defined as

$$\mathcal{P} = \{q \in \mathbb{R}^3 \mid n.(q - p) = 0\}.$$

- Now, for any point q in the plane

$$0 = n.(q - p) = (a, b, c).(x - x_0, y - y_0, z - z_0),$$

or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

- So another way to write the nonparametric equation of a plane, is

$$ax + by + cz = d,$$

where $d = ax_0 + by_0 + cz_0$. This is called the *point-normal equation* of the plane.

Hyperplanes.

- We saw a line in $\mathbb{R}^2$ can be written as

$$a_1x_1 + a_2x_2 = d$$

and a plane in $\mathbb{R}^3$ can be written in point-normal form as

$$a_1x_1 + a_2x_2 + a_3x_3 = d.$$

- Generalizing, a *hyperplane* in $\mathbb{R}^n$ can be written in point-normal form as

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = d,$$

where $(a_1, a_2, \ldots, a_n)$ is a normal.

  – The set of vectors in the hyperplane have tail at $(0, \ldots, 0, d/a_n)$ and are perpindicular to the normal vector to the hyperplane.

Example 1.

1. An economic application you have probably seen deals with commodity spaces.

   - The vector
   $$x = (x_1, x_2, \ldots, x_n)$$
   of nonnegative quantities of n commodities is called a *commodity bundle*. The set of all commodity bundles is the set
   $$\{(x_1, \ldots, x_n) \mid x_1 \geq 0, \ldots, x_n \geq 0\}$$
   and is called a *commodity space*.

   - Let $p_i > 0$ be the price of commodity i. The cost of buying bundle x is
   $$p_1x_1 + p_2x_2 + \cdots + p_nx_n = p \cdot x.$$
   A consumer with income I can purchase only bundles x for which $p \cdot x \leq I$. This subset of the commodity space is the consumer's *budget set*.

1. - The budget set is bounded above by the hyperplane $p \cdot x = I$, whose normal vector is the price vector p.

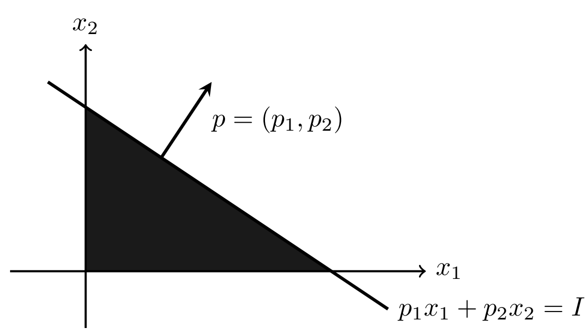

Figure 12: A consumer's budget set, $p \cdot x \leq I$, in commodity space.

2. • Another hyperplane you will see is the space of *probability vectors*

$$P_n = \{(p_1, \dots, p_2) \mid p_1 \geq 0, \dots, p_n \geq 0 \text{ and } p_1 + \cdots + p_n = 1\},$$

which is called the *probability simplex*. The probability simplex P_n is part of a hyperplane in $\mathbb{R}^n$ with normal vector $(1, 1, \dots, 1)$. ♦

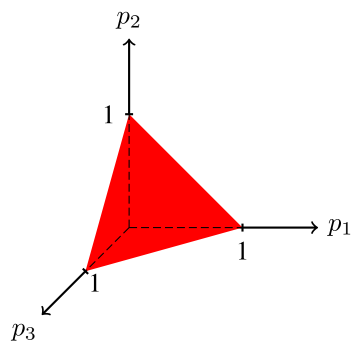

Figure 13: The probability simplex for $n = 3$.