# Set Theory

### Basic Definitions.

***Definition.*** A *set* is any collection of objects. Objects contained in the set are called the *elements* or *members* of the set.

- If $A$ is a set and $a$ is an element of $A$, we write $a \in A$.
- If $a$ is not contained in the set $A$, we write $a \notin A$.
- Given any set $A$ and any object $a$, we assume that precisely one of $a \in A$ or $a \notin A$ holds. ▲

- The simplest way of presenting a set is to list its elements. For example, the set consisting of the letters $a$, $b$, $c$, and $d$ is written

$$\{a, b, c, d\}.$$

- The order in which we list elements is irrelevant, so that the set $\{1, 2, 3\}$ is the same as the set $\{2, 3, 1\}$.
- We list each element of a set only once, so we would never write $\{1, 2, 2, 3\}$.
- There are four sets of numbers that we will use regularly.
  - The set of natural numbers

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

  - The set of integers

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

  - The set of rational numbers (also called fractions), denoted $\mathbb{Q}$. A real number $x$ is a *rational number* if there exist $n, m \in \mathbb{Z}$ such that $m \neq 0$ and $x = n/m$.
  - The set of real numbers, denoted $\mathbb{R}$.
- Sometimes we will also let $\mathbb{Z}_+$ denote the set of non-negative integers $\{0, 1, 2, \ldots\}$ and let $\mathbb{R}_{++}$ denote the set of positive real numbers. (Note there are no completely standard symbols for these sets).
- Another set we will see regularly is the *empty set* (or *null set*), which is the set that does not have any elements in it.
  - That is, the empty set is the set $\{\ \}$.
  - It is denoted $\emptyset$.

- Sometimes it is inconvenient or impossible to list explicitly all the elements of a set. Instead we present a set by describing it as the set of all the elements satisfying some criteria.
- For example, consider the set of all integers that are perfect squares.
  - We could write this set as

    $$S = \{n \in \mathbb{Z} \mid n \text{ is a perfect square}\},$$

    which is read "the set of all $n$ in $\mathbb{Z}$ such that $n$ is a perfect square."
  - To be more precise, we could write

    $$\begin{aligned} S &= \{n \in \mathbb{Z} \mid n = k^2 \text{ for some } k \in \mathbb{Z}\}, \text{ or} \\ S &= \{n \in \mathbb{Z} \mid \exists k \in \mathbb{Z} \text{ such that } n = k^2\}. \end{aligned}$$

- Examples of sets defined in this way are intervals in the real number line.
- Let $a, b \in \mathbb{R}$ be any two numbers. We then define the following sets.
  - Open interval:
    $$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}.$$
  - Closed interval:
    $$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}.$$
  - Half-open intervals:

    $$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\},$$

    $$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}.$$

- We can also define intervals which "go on forever".
  - Infinite intervals:
    $$[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\},$$
    $$(a, \infty) = \{x \in \mathbb{R} \mid a < x\},$$
    $$(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\},$$
    $$(-\infty, b) = \{x \in \mathbb{R} \mid x < b\},$$
    $$(-\infty, \infty) = \mathbb{R}.$$
- There are no intervals that are "closed" at $\infty$ or $-\infty$, for example, there is no interval of the form $[a, \infty]$.
- Since $\infty$ is not a real number it cannot be included in an interval contained in the real numbers.
- Intuitively, we have $A \subseteq B$ whenever the set $A$ is contained in the set $B$.

***Definition.*** Let $A$ and $B$ be sets. We say that $A$ is a *subset* of $B$ if $x \in A$ implies $x \in B$.

- If $A$ is a subset of $B$, we write $A \subseteq B$.
- If $A$ is not a subset of $B$, we write $A \nsubseteq B$. ▲

- Note "$A$ is a subset of $B$" is equivalent to "$B$ is a superset of $A$", which we write as $B \supseteq A$.

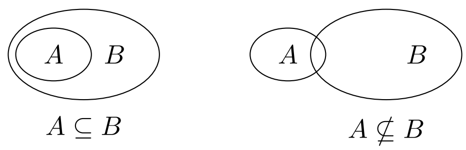


Figure 1: Definition of a subset.

- It is important to distinguish between
  - one thing being an element of a set, and
  - one set being a subset of another set.
- For example, suppose $A = \{a, b, c\}$. Then
  - $a \in A$ and $\{a\} \subseteq A$ are both true, whereas
  - "$a \subseteq A$" and "$\{a\} \in A$" are both false.
- Also, note that a set can be an element of another set. Suppose $B = \{\{a\}, b, c\}$. (Observe that $B$ is not the same set as $A$).
  - $\{a\} \in B$ and $\{\{a\}\} \subseteq B$ are both true, whereas
  - "$a \in B$" and "$\{a\} \subseteq B$" are both false.

**Theorem 1.** *Let $A$, $B$ and $C$ be sets.*

1. $A \subseteq A$.
2. $\emptyset \subseteq A$.
3. *If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

- Prove these! (Use the definition of a subset we just learnt).

***Definition.*** Let $A$ and $B$ be sets.

- We say that $A$ *equals* $B$, denoted $A = B$, if $A \subseteq B$ and $B \subseteq A$.
- We say that $A$ is a *proper subset* of $B$, denoted $A \subset B$, if $A \subseteq B$ and $A \neq B$. ▲

- Some texts write $A \subsetneq B$ to mean $A$ is a proper subset of $B$, while others use $A \subset B$ to mean what we write as $A \subseteq B$. Our notation makes $\subseteq$ and $\subset$ analagous to $\leq$ and $<$.

**Theorem 2.** *Let $A$, $B$ and $C$ be sets.*

1. *$A = A$.*
2. *If $A = B$ then $B = A$.*
3. *If $A = B$ and $B = C$, then $A = C$.*

- Prove these too!

***Definition.*** Let $A$ be a set. The *power set* of $A$ denoted $\mathcal{P}(A)$ (or sometimes $2^A$) is the set whose elements are the subsets of $A$. ▲

- In general, if a set has exactly $k$ elements, its power set will have exactly $2^k$ elements. (Why?)

  **Example 1.** Let $A = \{a, b, c\}$.

  – Then the subsets of $A$ are $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$ and $\{a, b, c\}$.
  – So, the power set is

  $$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

  – As expected the power set has eight elements. ♦

**Set Operations.**

***Definition.*** Let $A$ and $B$ be sets.

- The *union* of $A$ and $B$, denoted $A \cup B$, is the set defined by

  $$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

  It is the set containing everything that is either in $A$ or $B$ or both.
- The *intersection* of $A$ and $B$, denoted $A \cap B$, is the set defined by

  $$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

  It is the set containing everything that is in both $A$ and $B$. ▲

The following theorem establishes properties similar to those for addition and multiplication of numbers.

**Theorem 3.** *Let $A$, $B$, and $C$ be sets.*

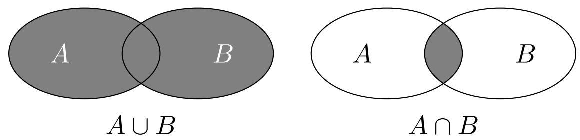


Figure 2: The union and intersection of sets $A$ and $B$.

1. *$A \cap B \subseteq A$ and $A \cap B \subseteq B$. If $X$ is a set such that $X \subseteq A$ and $X \subseteq B$, the $X \subseteq A \cap B$.*
2. *$A \subseteq A \cup B$ and $B \subseteq A \cup B$. If $Y$ is a set such that $A \subseteq Y$ and $B \subseteq Y$, the $A \cup B \subseteq Y$.*
3. *$A \cup B = B \cup A$ and $A \cap B = B \cap A$ (Commutative Laws).*
4. *$(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$ (Associative Laws).*
5. *$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive Laws).*
6. $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$ (Identity Laws).
7. $A \cup A = A$ and $A \cap A = A$ (Idempotent Laws).
8. $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$ (Absorption Laws).
9. If $A \subseteq B$, then $A \cup C \subseteq B \cup C$ and $A \cap C \subseteq B \cap C$.

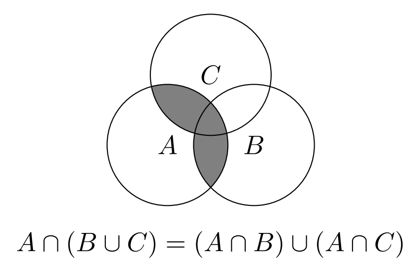


Figure 3: Distributivity of the union over the intersection.

*Proof of (5).* We prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, the other proof is similar.

- Let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$, i.e. $x \in B$ or $x \in C$.
  - In the first case we have $x \in A \cap B$.
  - In the second case we have $x \in A \cap C$.
  - It follows that in either case we have $x \in (A \cap B) \cup (A \cap C)$. Thus $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

- Now let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$.
  - In the first case, $x \in A$ and $x \in B$, and it follows that $x \in B \cup C$.
  - In the second case, $x \in A$ and $x \in C$, and it follows that $x \in B \cup C$.
  - In either case we have $x \in A \cap (B \cup C)$. Thus $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.
- Combining this result with that of the preceding paragraph we have shown that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. ■

***Definition.*** Let $A$ and $B$ be sets.

- We say that the sets $A$ and $B$ are *disjoint* if $A \cap B = \emptyset$.
- Let $A$ and $B$ be sets. The *difference* (or *set difference*) of $A$ and $B$, denoted $A \setminus B$ or $A - B$ is the set defined by

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

- The *symmetric difference* of $A$ and $B$, denoted $A \triangle B$ is the set defined by

$$A \triangle B = \{x \mid (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}. \quad ▲$$

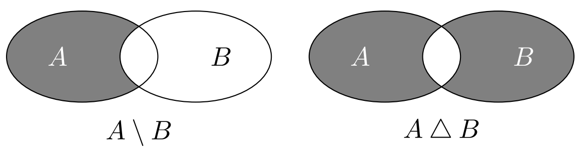


Figure 4: The difference and symmetric difference of sets $A$ and $B$.

**Theorem 4.** *Let $A$, $B$, and $C$ be sets.*

1. $A \setminus B \subseteq A$.
2. $(A \setminus B) \cap B = \emptyset$.
3. $A \setminus B = \emptyset$ *iff* $A \subseteq B$.
4. $B \setminus (B \setminus A) = A$ *iff* $A \subseteq B$.
5. *If* $A \subseteq B$, *then* $A \setminus C = A \cap (B \setminus C)$.
6. *If* $A \subseteq B$, *then* $C \setminus A \supseteq C \setminus B$.
7. $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$ *and* $C \setminus (A \cap B) = (C \setminus A) \cup (C \setminus B)$ *(De Morgan's Laws).*

- We denote an *ordered pair* of elements as $(a, b)$. As the name suggests, the order of the elements matters, i.e. the ordered pair $(a, b)$ equals the ordered pair $(a', b')$ *iff* $a = a'$ and $b = b'$.

***Definition.*** Let $A$ and $B$ be sets. The *(Cartesian) product* of $A$ and $B$, denoted $A \times B$, is the set

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

where $(a, b)$ denotes an ordered pair. ▲

- We often write $A^2$ for $A \times A$.
- Note that the Cartesian product of two sets is not commutative.
  - For example, let $\{a\}$ and $\{b\}$ be sets where $a$ and $b$ are two distinct objects. Then
  $$\{a\} \times \{b\} = \{(a, b)\} \neq \{(b, a)\} = \{b\} \times \{a\}.$$
- Formally speaking, it is not associative either, since $(a, (b, c))$ is not the same thing as $((a, b), c)$. But there is a natural correspondence between the elements of $A \times (B \times C)$ and $(A \times B) \times C$ so we can think of these sets as the same and simply refer to $A \times B \times C$.

  ***Definition.*** Let $A_1, \dots, A_n$ (for any $n \in \mathbb{N}$) be sets. We define an *$n$-vector* as a list $(a_1, \dots, a_n)$, where $a_i \in A_i$ for each $i = 1, \dots, n$. The *(Cartesian) product*) of $n$ sets $A_1, \dots, A_n$, denoted $A_1 \times \cdots \times A_n$, is the set

  $$A_1 \times \cdots \times A_n = \{(a_1, \dots, a_n) \mid a_i \in A_i,\ i = 1, \dots, n\}.$$ ▲

- We often write $\prod_{i=1}^{n} A_i$ to denote $A_1 \times \cdots \times A_n$, and refer to this as the *$n$-fold product* of $A_1, \dots, A_n$. If $A_i = A$ for all $i$, we then write $A^n$.

**Theorem 5.** *Let $A$, $B$, $C$ and $D$ be sets.*

1. *If $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$.*
2. *$A \times (B \cup C) = (A \times B) \cup (A \times C)$ and $(B \cup C) \times A = (B \times A) \cup (C \times A)$ (Distributive Laws).*
3. *$A \times (B \cap C) = (A \times B) \cap (A \times C)$ and $(B \cap C) \times A = (B \times A) \cap (C \times A)$ (Distributive Laws).*
4. *$A \times \emptyset = \emptyset$ and $\emptyset \times A = \emptyset$.*
5. *$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.*

- We can depict statement (5) of the theorem in a diagram.
- We can see that a similar statement does not hold for the union i.e.

$$(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D).$$

$(A \cup B) \times (C \cup D)$

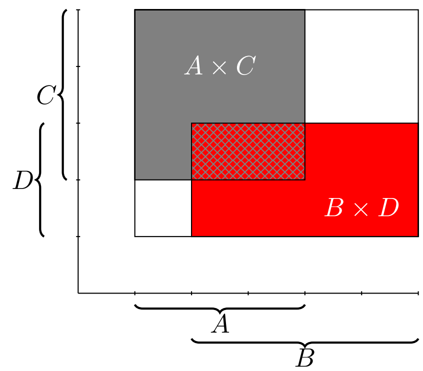


Figure 5: The expression in statement (5) is represented by the crosshatched area.

**Indexed Families of Sets.**

- So far we have only looked at the union and intersection of two sets at a time.
- What about more than two sets? For finitely many sets we can use the definitions we already have, making use of the associative laws.
- Suppose, for example, we had three sets $A$, $B$ and $C$, and we wanted to form the union of all three of them.
  - Since our original definition works for only two sets at a time, we could form the union of all three sets in one of two possible ways, namely $(A \cup B) \cup C$ and $A \cup (B \cup C)$.
  - But by associativity $(A \cup B) \cup C = A \cup (B \cup C)$, so we denote either expression as $A \cup B \cup C$.
  - The union operation is commutative, so we can also change the order of the three sets. So altogether there are twelve equivalent ways to form their union.
- The same idea holds for intersections of three sets and for any finite collection of sets $A_1, A_2, \ldots, A_n$ (where we emphasize that the word "finite" applies to the number of sets, not the sizes of the individual sets).
- For ease of notation we use an analogue of the summation notation and write

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n,$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n.$$

**Indexed Families of Sets.**

- What about unions and intersections of infinite collections of sets? We would also like to define the infinite union and intersection $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$.
  - We could try to form $A_1 \cup A_2$, then $A_1 \cup A_2 \cup A_3$, and so on, then take the "limit". But there is a simpler way to proceed.
- We use the definitions of $A \cup B$ and $A \cap B$ as models and define the union and intersection of an infinite collection of sets directly as

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots = \{x \mid x \in A_n \text{ for some } n \in \mathbb{N}\},$$
$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots = \{x \mid x \in A_n \text{ for all } n \in \mathbb{N}\}.$$

- Sometimes $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ are written as $\bigcup_{i\in\mathbb{N}} A_i$ and $\bigcap_{i\in\mathbb{N}} A_i$.

**Example 2.**

1. For each $i \in \mathbb{N}$, let $B_i$ be the set $B_i = \{1, 2, \ldots, 3i\}$. Then $\bigcup_{i=1}^{\infty} B_i = \{1, 2, \ldots\}$ and $\bigcap_{i=1}^{\infty} B_i = \{1, 2, 3\}$.
2. For each $k \in \mathbb{N}$, let $F_k$ be the open interval $F_k = (1/k, 8 + 3/k)$. Then $\bigcup_{k=1}^{\infty} F_k = (0, 11)$ and $\bigcap_{k=1}^{\infty} F_k = (1, 8]$. ♦

- We need unions and intersections for more general situations than those above.
  - Suppose, for example, that for each real number $x$ we define the set $Q_x$ to be all real numbers less than $x$, i.e. $Q_x = (-\infty, x)$.
  - It turns out, as we will see later, that there is no possible way to line up all the sets of the form $Q_x$ in order analogously to $A_1, A_2, A_3, \ldots$.
- This leads us to the following definition, in which we do not need to think of the sets written in order. Instead we can think of the collection of sets as having one set for each element of a set $I$.

***Definition.*** Let $I$ be a non-empty set. Supppose there is a set denoted $A_i$ for each element $i \in I$. Such a collection is called a *family of sets indexed by* $I$; the set $I$ is called the *indexing set* for this family of sets. We denote the family of sets by

$$\{A_i\}_{i\in I}.$$

The *union* and *intersection* of all the sets in the family of sets are defined by

$$\bigcup_{i\in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\},$$
$$\bigcap_{i\in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}.$$
▲

**Example 3.**

1. For each $x \in \mathbb{R}$, let $C_x$ be the interval of real numbers $C_x = [-x/3, x/2]$. Then $\bigcup_{x\in\mathbb{R}} C_x = \mathbb{R}$ and $\bigcap_{x\in\mathbb{R}} C_x = \{0\}$.
2. Let $I$ be the set $I = \{a, \{b, c\}, d, \emptyset\}$. Let
   - $X_a = \{1, 3, 6\}$,
   - $X_{\{b,c\}} = \{2, 4\}$
   - $X_d = \{5, 8, 10, 11\}$, and
   - $X_\emptyset = \{7, 9\}$.

   Then $\bigcup_{i\in I} X_i = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ and $\bigcap_{i\in I} X_i = \emptyset$. ⧫

**Theorem 6.** *Let $I$ be a non-empty set, let $\{A_i\}_{i\in I}$ be a family of sets indexed by $I$, and let $B$ be a set.*

1. $A_k \subseteq \bigcup_{i\in I} A_i$ *for all* $k \in I$. *If* $A_k \subseteq B$ *for all* $k \in I$, *then* $\bigcup_{i\in I} A_i \subseteq B$.
2. $\bigcap_{i\in I} A_i \subseteq A_k$ *for all* $k \in I$. *If* $B \subseteq A_k$ *for all* $k \in I$, *then* $B \subseteq \bigcap_{i\in I} A_i$.
3. $B \cap \left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I}(B \cap A_i)$.
4. $B \cup \left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I}(B \cup A_i)$.
5. $B \setminus \left(\bigcup_{i\in I} A_i\right) = \bigcap_{i\in I}(B \setminus A_i)$.
6. $B \setminus \left(\bigcap_{i\in I} A_i\right) = \bigcup_{i\in I}(B \setminus A_i)$.

- Part (i) says that $\bigcup_{i\in I} A_i$ is the smallest set containing all the sets $A_k$, and part (ii) says that $\bigcap_{i\in I} A_i$ is the largest set contained in all the sets $A_k$.
- I will prove (iii). Prove the rest for yourself!