Sequences

Limits of Sequences.

Definition. A real-valued *sequence* s is any function $s : \mathbb{N} \to \mathbb{R}$. ▲

- Usually, instead of using the notation $s(n)$, we write s_n for the value of this function calculated at n.
- We also commonly write a sequence s as $(s_n)_{n\in\mathbb{N}}$, $(s_n)_{n=1}^{+\infty}$ or simply (s_n).
- The *image* of a sequence (s_n) is the set $\{s_n | n \in \mathbb{N}\}$.

Example 1.

- (s_n) given by $s_n = \frac{1}{n}$ for all $n \in \mathbb{N}$ is the sequence

$$(1, 1/2, 1/3, 1/4, 1/5, \ldots)$$

- (x_n) given by $x_n = (-1)^n$ for all $n \in \mathbb{N}$ is the sequence

$$(-1, 1, -1, 1, -1, \ldots).$$

  Here, x is a function with domain $\{1, 2, \ldots\}$ and image $\{-1, 1\}$.

- (a_n) given by $a_n = \left(1 + \frac{1}{n}\right)^n$ for all $n \in \mathbb{N}$ is the sequence

$$(2, 2.25, 2.3704, 2.4414, 2.4883, 2.5216, \ldots).$$

  (Can anyone guess what happens as $n \to \infty$?) ♦

Definition. A real valued sequence (s_n) is said to *converge* to the real number s if

> for each $\varepsilon > 0$ there exists a number N, such that
> $n > N$ implies that $|s_n - s| < \varepsilon$.

- If (s_n) converges to s we write $\lim_{n\to\infty} s_n = s$, $\lim s_n = s$ or $s_n \to s$.
- s is called the *limit* of (s_n).
- A sequence that does not converge to some real number is said to *diverge*. ▲
- In general, the value of N depends on ε, and sometimes, to emphasize this fact, we may write $N(\varepsilon)$ or N_ε.

Theorem 1. *A convergent sequence (s_n) has a unique limit.*

Proof[1]. By way of contradiction, assume that (s_n) has two distinct limits, s and t. Consider any $\varepsilon > 0$. By definition of a limit, there must exist a number N_1 such that

$$n > N_1 \Rightarrow |s_n - s| < \frac{\varepsilon}{2},$$

and there must exist a number N_2 such that

$$n > N_2 \Rightarrow |s_n - t| < \frac{\varepsilon}{2}.$$

But then, for $n > \max\{N_1, N_2\}$, the triangle inequality implies that

$$|s - t| = |(s - s_n) + (s_n - t)| \leq |s - s_n| + |s_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $|s-t| < \varepsilon$ for every $\varepsilon > 0$, implying that $s = t$, a contradiction to the assumption that s and t were distinct. ∎

Example 2. Show that $\lim 1/n^p = 0$ for $p > 0$.

- We need to find, for every $\varepsilon > 0$, an N such that $|1/n^p - 0| < \varepsilon$ for all $n > N$.
- Since $n > 0$, we need an N such that $1/n^p < \varepsilon$ for $n > N$. Solving for n we see that we need $n > (1/\varepsilon)^{\frac{1}{p}}$.

Proof. Consider any $\varepsilon > 0$. Now let $N = (1/\varepsilon)^{\frac{1}{p}}$. Then $n > N$ implies $1/n^p < \varepsilon$. ∎

Example 3. $\lim \frac{4n^2+n+1}{n^3+5} = 0$.

- We need to find an upper bound for the numerator and a lower bound for the denominator.
- We can see that $4n^2 + n + 1 \leq 6n^2$ and that $n^3 + 5 \geq n^3$ for all $n \in \mathbb{N}$. So we need $6n^2/n^3 = 6/n < \varepsilon$ or $n > 6/\varepsilon$.

Proof. Let $\varepsilon < 0$ and let $N = 6/\varepsilon$. Then if $n > N$ we have $n > 6/\varepsilon$ or $6/n < \varepsilon$. Now

$$\frac{4n^2 + n + 1}{n^3 + 5} \leq \frac{6n^2}{n^3} = \frac{6}{n} < \varepsilon.$$

So for $n > N$ we have

$$\left| \frac{4n^2 + n + 1}{n^3 + 5} \right| < \varepsilon.$$ ∎

[1] For an alternative proof, see S&B theorem 12.1.

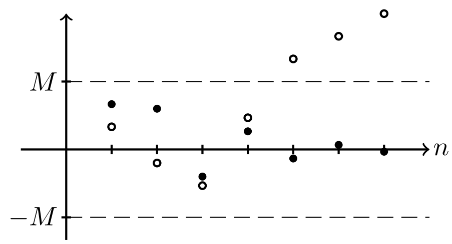

Figure 1: The sequence represented by the filled circles is bounded. The other sequence is bounded below, but not bounded above.

Limit Theorems for Sequences.

Definition. Let S be a nonempty subset of $\mathbb{R}$.

- If a real number M satisfies $s \leq M$ for all $s \in S$, then M is called an *upper bound* of S, and S is said to be *bounded above*.
- If a real number m satisfies $m \leq s$ for all $s \in S$, then m is called a *lower bound* of S, and S is said to be *bounded below*.
- The set S is *bounded* if it is bounded above and bounded below.
- If S is bounded above and S has a least upper bound, then we call it the *supremum* of S, denoted $\sup S$.
- If S is bounded above and S has a greatest lower bound, then we call it the *infimum* of S, denoted $\inf S$. ▲

Limit Theorems for Sequences.

Definition. Let (s_n) be a real-valued sequence and let $S = \{s_n | n \in \mathbb{N}\}$ be its image.

- (s_n) is said to be *bounded above* if the set S is bounded above.
- (s_n) is said to be *bounded below* if the set S is bounded below.
- (s_n) is said to be *bounded* if the set S is a bounded set, i.e. if there exists a constant M such that $|s_n| \leq M$ for all n. ▲

Theorem 2 (Ross 9.1)**.** *Convergent sequences are bounded.*

Proof. Let (s_n) be a convergent sequence with limit s. Let $\varepsilon = 1$. By definition of a limit, there exists $N \in \mathbb{N}$ such that

$$n > N \Rightarrow |s_n - s| < 1$$

From the triangle inequality, $|s_n| = |s + (s_n - s)| \leq |s| + |s_n - s|$. Thus, we have $n > N$ implies $|s_n| < |s| + 1$. Let $M = \max\{|s| + 1, |s_1|, |s_2, \ldots < |s_N|\}$. Then $|s_n| \leq M$ for all $n \in \mathbb{N}$ and so (s_n) is a bounded sequence. ■

- This theorem gives a simple way of showing certain sequences are not convergent – check whether they are bounded.
- The converse of the theorem does not hold. It is *not* true that all bounded sequences converge, as the following example shows.

Example 4. The bounded sequence $a_n = (-1)^n$ does not converge.

Proof. By way of contradiction, assume that (a_n) *does* converge to a limit a. Let $\varepsilon = 1$. Then, by the definition of a limit, there exists N such that $n > N$ implies $|(-1)^n - a| < 1$. Thus, for even and odd $n > N$ we have that $|1 - a| < 1$ and $|-1 - a| < 1$, respectively. Now, by the triangle inequality

$$2 = |1 - (-1)| = |1 - a + a - (-1)| \leq |1 - a| + |a - (-1)|.$$

But then $2 \leq |1 - a| + |-1 - a| < 1 + 1 = 2$, an absurdity. ■

- The following limit theorems allow us to easily calculate many limits.

Theorem 3 (Ross 9.2-9.6)**.** *Suppose that (s_n) converges to s, that (t_n) converges to t and $k \in \mathbb{R}$.*

1. $\lim(ks_n) = k \lim s_n$.
2. $\lim(s_n + t_n) = \lim s_n + \lim t_n$.
3. $\lim(s_n t_n) = (\lim s_n)(\lim t_n)$.
4. $\lim 1/s_n = 1/s$, *if $s \neq 0$ and $s_n \neq 0$ for all n.*
5. $\lim(t_n/s_n) = t/s$, *if $s \neq 0$ and $s_n \neq 0$ for all n.*
6. *If there exists an N such that $s_n \leq t_n$ for all $n > N$, then $s \leq t$.*
7. *If $s = t$ and there exists an N such that $s_n \leq r_n \leq t_n$ for all $n > N$, then r_n converges to s.*

Example 5.

- Find $\lim s_n$ where $s_n = (n - 5)/(n^2 + 7)$. We can rewrite s_n as

$$s_n = \frac{\frac{1}{n} - \frac{5}{n^2}}{1 + \frac{7}{n^2}}.$$

Now $\lim(1/n - 5/n^2) = 0$ by claim 2 and limit theorems (1) and (2). Similarly, $\lim(1 + 7/n^2) = 1$. So by limit theorems (3) and (4), $\lim s_n = 0/1 = 0$.

- Find $\lim t_n$ where $t_n = (n^2+3)/(n+1)$. We can rewrite t_n as

$$\frac{n+\frac{3}{n}}{1+\frac{1}{n}} \quad \text{or} \quad \frac{1+\frac{3}{n^2}}{\frac{1}{n}+\frac{1}{n^2}},$$

  but both fractions give us problems: either the numerator does not converge, or the denominator converges to 0. Turns out t_n does not converge, it goes to $+\infty$. We now define precisely what we mean by $\lim t_n = +\infty$. ⧫

Definition. Let (s_n) be a real valued sequence.

- (s_n) *diverges to* $+\infty$ if

  for each $M > 0$ there is a number N such that
  $n > N$ implies $s_n > M$.

  In this case, we write $\lim s_n = +\infty$.

- Similarly, (s_n), *diverges to* $-\infty$ if

  for each $M < 0$ there is a number N such that
  $n > N$ implies $s_n < M$.

  In this case, we write $\lim s_n = -\infty$.

- We say that (s_n) has a *limit* or that the *limit exists* provided that (s_n) converges, diverges to $+\infty$, or diverges to $-\infty$. ▲

- Many sequences do not have limits $+\infty$ or $-\infty$ even if they are unbounded, e.g. $s_n = n(-1)^n$ does not diverge to $+\infty$ or $-\infty$.

Example 6. $\lim[(n^2+3)/(n+1)] = +\infty$.

- Consider $M > 0$. We need to find how large n must be to ensure $(n^2+3)/(n+1) > M$. We will bound the numerator from below and the denominator from above.

- We have $n^2+3 > n^2$ and $n+1 \leq 2n$, from which it follows that $(n^2+3)/(n+1) > n^2/2n = n/2$. So we need $n/2 > M$.

Proof. Let $M > 0$, and let $N = 2M$. Then $n > N$ implies $n/2 > M$ and so we have

$$\frac{n^2+3}{n+1} > \frac{n^2}{2n} = \frac{n}{2} > M.$$

It follows that $\lim[(n^2+3)/(n+1)] = +\infty$. ■

- The first theorem concerning infinite limits is useful for computing the limit of a sequence which can be written as the ratio of two other sequences with certain properties.

Theorem 4 (Ross 9.9)**.** *Let* (s_n) *and* (t_n) *be sequences such that* $\lim s_n = +\infty$ *and* $\lim t_n > 0$. *Then* $\lim s_n t_n = +\infty$.

Example 7. We can use this to prove $\lim[(n^2+3)/(n+1)] = +\infty$. Let $s_n = n+3/n$ and let $t_n = 1/(1+1/n)$. It is easy to show (try – use the limit theorems) $\lim s_n = +\infty$ and $\lim t_n = 1$. So by the theorem we have $\lim[(n^2+3)/(n+1)] = +\infty$.

- The second theorem says that a sequence diverges to $+\infty$ iff the sequence with reciprocal terms converges to 0.

Theorem 5 (Ross 9.10)**.** *For a sequence* (s_n) *of positive real numbers, we have* $\lim s_n = +\infty$ *iff* $\lim(1/s_n) = 0$.

Example 8. The sequence given by $s_n = n$ clearly diverges to $+\infty$, while the sequence given by $t_n = 1/n$ converges to 0.

Monotone Sequences and Cauchy Sequences.

Definition. Let (s_n) be a real valued sequence.

- (s_n) is *nondecreasing* if $s_n \leq s_{n+1}$ for all n.
- (s_n) is *nonincreasing* if $s_n \geq s_{n+1}$ for all n.
- (s_n) is *monotone* if it is nondecreasing or nonincreasing. ▲

Example 9. Consider the sequences defined, for all $n \in \mathbb{N}$, by

- $a_n = 1 - 1/n$,
- $b_n = n^3$,
- $c_n = (1+1/n)^n$,
- $d_n = 1/n^2$
- $s_n = (-1)^n$
- $t_n = \cos(n\pi/3)$
- $u_n = (-1)^n n$,
- $v_n = (-1)^n/n$.

Then (a_n), (b_n) and (c_n) are nondecreasing sequences, while (d_n) is nonincreasing. The sequences (s_n), (t_n), (u_n) and (v_n) are not monotonic sequences. The sequences (a_n), (c_n), (d_n), (s_n), (t_n), and (v_n) are bounded sequences, while (b_n) and (u_n) are unbounded. ♦

Theorem 6 (Ross 10.2)**.** *All bounded monotone sequences converge.*

Proof. We will show that all bounded nondecreasing sequences converge, the proof for nonincreasing sequences being similar.

- Let (s_n) be a bounded nondecreasing sequence.
- Let S denote the image of the sequence, i.e. the set $\{s_n \mid n \in \mathbb{N}\}$, and let $u = \sup S$. By definition of a bounded sequences, the set S is bounded, and u represents a real number.

- We will show that $\lim s_n = u$.

- Let $\varepsilon > 0$. Since $u - \varepsilon (< u)$ is not an upper bound for S, there exists an N such that $s_N > u - \varepsilon$.
- Now, since (s_n) is nondecreasing, we have $s_N \leq s_n$, for all $n > N$.
- Furthermore, $s_n \leq u$ for all n (since u is an upper bound of S).
- Thus $n > N$ implies $u - \varepsilon < s_n \leq u$, which implies $|s_n - u| < \varepsilon$.

This shows that $\lim s_n = u$. ■

- This proof relies on the Completeness Axiom, which says that every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound. That is $\sup S$ exists and is a real number.

Theorem 7 (Ross 10.4)**.**

1. *If* (s_n) *is an unbounded nondecreasing sequence, then* $\lim s_n = +\infty$.
2. *If* (s_n) *is an unbounded nonincreasing sequence, then* $\lim s_n = -\infty$.

Proof of (i). Let (s_n) be an unbounded nondecreasing sequence and let $M > 0$.

- Since the set $\{s_n \mid n \in \mathbb{N}\}$ is unbounded and it is bounded below by s_1, it must be unbounded above. Thus, for some $N \in \mathbb{N}$, we have $s_N > M$.

Then $n > N$ implies $s_n \geq s_N > M$ and so by definition (14) $\lim s_n = +\infty$. ■

- Try the proof of part (2) yourself.

Definition. A real valued sequence (s_n) is called a *Cauchy sequence* if

for each $\varepsilon > 0$ there exists a number N, such that
$m, n > N$ implies that $|s_n - s_m| < \varepsilon$.

Compare this with the definition of a convergent sequence. ▲

Lemma 1 (Ross 10.9)**.** *Convergent sequences are Cauchy sequences.*

Proof. Let $s = \lim s_n$. The idea of the proof is that, since the terms s_n are close to s for large n, they must also be close to each other.

- Let $\varepsilon > 0$. Then there exists an N such that

$$n > N \text{ implies } |s_n - s| < \frac{\varepsilon}{2} \text{ and } m > N \text{ implies } |s_m - s| < \frac{\varepsilon}{2}.$$

- Thus, $m, n > N$ implies

$$|s_n - s_m| = |s_n - s - (s_m - s)| \leq |s_n - s| + |s - s_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

where we have used the triangle inequality.

Thus (s_n) is a Cauchy sequence. ■

- With a bit more work we can show that the converse is true.

Lemma 2. *Cauchy sequences are covergent sequences.*

- The following theorem is important. It enables us to verify that a sequence converges by checking that it is a Cauchy sequence. It follows directly from the previous two lemmas.

Theorem 8 (Ross 10.11)**.** *A sequence is a convergent sequence iff it is a Cauchy sequence.*

Subsequences.

- Next we define a subsequence, which is a sequence made up of a selection of some (or possibly all) of the terms of the original sequence taken in order.

Definition. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. A *subsequence* of this sequence is a sequence of the form $(t_k)_{k\in\mathbb{N}}$, where for each k there is a natural number n_k such that

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots \tag{1}$$

$$\text{and} \quad t_k = s_{n_k}. \tag{2}$$ ▲

- Note that (1) defines an infinite subset of $\mathbb{N}$, i.e. $\{n_1, n_2, \ldots\}$, so a subsequence of (s_n) is a sequence obtained by selecting (in order) an infinite subset of the terms.

- We could write an alternative definition, using composition.
  - Let $\sigma : \mathbb{N} \to \mathbb{N}$ be an increasing function given by $\sigma(k) = n_k$ for $k \in \mathbb{N}$.
  - The function σ selects an infinite subset of $\mathbb{N}$ in order.
  - The subsequence of s corresponding to σ is just the composite function $t = s \circ \sigma$.

    $$t_k = t(k) = (s \circ \sigma)(k) = s(\sigma(k)) = s(n_k) = s_{n_k}$$

    for all $k \in \mathbb{N}$.

- When we say "a subsequence (s_{n_k}) of (s_n)" we mean a subsequence defined as above.

Example 10. Let (s_n) be the sequence defined by $s_n = (-2)^n$ for all $n \in \mathbb{N}$. The negative terms of this sequence make up a subsequence.

- The sequence (s_n) is
$$(-2, 4, -8, 16, -32, 64, -128, 256, -512, 1024, \ldots)$$
and the subsequence is
$$(-2, -8, -32, -128, -512, \ldots).$$
- More precisely, the subsequence is $(s_{n_k})_{k\in\mathbb{N}}$, where $n_k = 2k - 1$ for all $k \in \mathbb{N}$. So $s_{n_k} = (-2)^{2k-1}$.
- The selection function σ is given by $\sigma(k) = 2k - 1$ for all $k \in \mathbb{N}$. ♦

Theorem 9 (Ross 11.2)**.** *If the sequence (s_n) converges, then every subsequence converges to the same limit.*

Proof. Let (s_{n_k}) denote a subsequence of (s_n). We will show $\lim s_{n_k} = s$.

- First note that $n_k \geq k$ for all k. We can prove this by induction.
  - It is true for $k = 1$, since $n_1 \geq 1$.
  - Suppose it is true for k i.e. suppose that $n_k \geq k$.
  - By definition of a subsequence, $n_{k+1} > n_k$, and so using our inductive hypothesis $n_{k+1} > k$. Hence, $n_{k+1} \geq k+1$, and it follows, by the principle of mathematical induction, that $n_k \geq k$ for all k.
- Let $s = \lim s_n$, and let $\varepsilon > 0$. Then, by definition, there exists an N such that $n > N$ implies $|s_n - s| < \varepsilon$.
- Now, $k > N$ implies $n_k > N$, which implies $|s_{n_k} - s| < \varepsilon$.
- Thus $\lim s_{n_k} = s$. ■

Theorem 10 (Ross 11.3)**.** *Every sequence (s_n) has a monotonic subsequence.*

Theorem 11 (Bolzano-Weierstrass)**.** *Every bounded sequence has a convergent subsequence.*

- There are different important equilibrium concepts in economics, the proofs of the existence of which often require variations of the Bolzano-Weierstrass theorem.
- One example is the existence of a Pareto efficient allocation. An allocation is a matrix of consumption bundles for agents in an economy, and an allocation is pareto efficient if no change can be made to it which makes no agent worse off and at least one agent better off. The Bolzano-Weierstrass theorem allows one to prove that if the set of allocations is compact and non-empty, then the system has a Pareto efficient allocation.