MTAEA – Relations

Scott McCracken

School of Economics,
Australian National University

January 13, 2010

Relations.

Definition.

Let A and B be sets.

- A relation R from A to B is a subset $R \subseteq A \times B$.
- If $a \in A$ and $b \in B$, we write $a\,R\,b$ if $(a, b) \in R$, and $a \not R\, b$ if $(a, b) \notin R$.
- A relation from A to A is called a relation on A. ▲

- So, if $(x, y) \in R$ or $(y, x) \in R$, we then think of R as associating the object x with y. If $\{(x, y), (y, x)\} \cap R = \emptyset$ then there is no connection between x and y embodied in R.

Relations.

Example

(1) Let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$. There are many possible relations from A to B, one example of which is the relation S defined by $S = \{(1, y), (1, z), (2, y)\}$. Then $1\,S\,y$, $1\,S\,z$, and $2\,S\,y$.

(2) The symbols $<$ and $\leq$ both represent relations on $\mathbb{R}$.

(3) Let $P = \{\Bicycle, \Football, \Letter\}$ and let H be the relation on P defined by having an object x related to an object y iff x is heavier than y. Then $H = \{(\Bicycle, \Football), (\Bicycle, \Letter), (\Football, \Letter)\}$ or $\Bicycle\,H\,\Football$, $\Bicycle\,H\,\Letter$, and $\Football\,H\,\Letter$.

(4) Let A be a set. Define a relation on $\mathcal{P}(A)$ by saying that $P, Q \in \mathcal{P}(A)$ are related iff $P \subseteq Q$.

(5) In economics, the preference-based approach summarizes the objectives of a decision maker in a preference relation on the set of alternatives X. This relation is denoted by $\succsim$ and, for any $x, y \in X$, we read $x \succsim y$ as "x is at least as good as y". $\blacklozenge$

Relations.

Definition.

Let A and B be non-empty sets, and let R be a relation from A to B. For each $x \in A$, define the relation class of x with respect to R, denoted $R[x]$, to be the set

$$R[x] = \{y \in B \mid x \, R \, y\}$$

We will sometimes write $[x]$ instead of $R[x]$ if the relation R is understood from the context. ▲

Example

We continue examples (1) and (2) from above.

(1) For this relation we have $[1] = \{y, z\}$, we have $[2] = \{y\}$ and we have $[3] = \emptyset$.

(2) For the relation $<$, we have $[x] = (x, \infty)$ for all $x \in \mathbb{R}$, and for the relation $\leq$, we have $[x] = [x, \infty)$ for all $x \in \mathbb{R}$. ♦

Relations.

Definition.

Let A be a non-empty set and let R be a relation on A.

- The relation R is complete if either $x\,R\,y$ or $y\,R\,x$ holds for all $x, y \in A$.
- The relation R is reflexive if $x\,R\,x$ for all $x \in A$.
- The relation R is irreflexive if $x \not R\, x$ for all $x \in A$.
- The relation R is symmetric if $x\,R\,y$ implies $y\,R\,x$ for all $x, y \in A$.
- The relation R is asymmetric if $x\,R\,y$ implies $y \not R\, x$ for all $x, y \in A$.
- The relation R is antisymmetric if $x\,R\,y$ and $y\,R\,x$ imply $x = y$ for all $x, y \in A$.
- The relation R is transitive if $x\,R\,y$ and $y\,R\,z$ imply $x\,R\,z$ for all $x, y, z \in A$. ▲

Relations.

Example

(1) Let $B = \{x, y, z\}$, and let $T = \{(x, x), (y, y), (z, z), (x, y), (y, z)\}$. Then T is reflexive, but neither symmetric nor transitive. It is reflexive because (x, x), (y, y) and (z, z) are all in T. The relation is not symmetric, since $x\, T\, y$ but $y \not T x$. The relation is not transitive, since $x\, T\, y$ and $y\, T\, z$ but $x \not T z$.

(2) The relation $<$ on $\mathbb{R}$ is transitive, but is not symmetric or reflexive. Let $x, y, z \in \mathbb{R}$. If $x < y$ and $y < z$, then $x < z$, and so the relation is transitive. The relation is not symmetric since if $x < y$, it is never true that $y < x$. It is not reflexive because we never have $x < x$.

(3) The relation $\leq$ on $\mathbb{R}$ is complete, reflexive (so not irreflexive), transitive and antisymmetric (but not symmetric).

(4) A preference relation $\succsim$ is a rational preference relation if it is both transitive and complete. ♦

Relations.

- We can also create new relations from old.

Definition.

Let A be a set and let R be a reflexive relation on A.

- The asymmetric part of R is the relation P_R on A, defined by $x \, P_R \, y$ iff $x \, R \, y$ and $y \not R \, x$.
- The symmetric part of R is the relation I_R on A, defined by $x \, I_R \, y$ iff $x \, R \, y$ and $y \, R \, x$.

▲

Example

For the preference relation $\succsim$ on a set of alternatives X, the asymmetric part is denoted $\succ$ and $x \succ y$ is read "x is preferred to y". The symmetric part is denoted $x \sim y$ and $x \sim y$ is read "x is indifferent to y".

♦

Relations.

Example

Another important type of relation on the integers is related to the idea of "clock arithmetic".

- Suppose it is 2 o'clock, and we want to know what time it will be in 3 hours. Clearly it will be $2+3=5$ o'clock.
- Now suppose it is 7 o'clock, and you want the time it will be in 7 hours. Proceeding as above would give $7+7=14$ o'clock, but the correct answer is obviously 2 o'clock, which we get by subtracting 12 from 14, because 14 is greater than 12.

We can generalize this idea.

Definition.

Let $n \in \mathbb{N}$. If $a, b \in \mathbb{Z}$, we say that a is congruent to b modulo n, denoted $a \equiv b \pmod{n}$ if $a - b = kn$ for some $k \in \mathbb{Z}$. ▲

- For each fixed $n \in \mathbb{N}$, we obtain a relation on $\mathbb{Z}$ given by congruence modulo n. ♦

Equivalence Relations.

Definition.

Let A be a set and let $\sim$ be a relation on A. The relation $\sim$ is an **equivalence relation** if it is reflexive, symmetric and transitive. ▲

Example

The following are equivalence relations.

(1) Equality on the set $\mathbb{R}$.

(2) Being the same age on the set of all people.

(3) For rational preferences, the indifference relation $\sim$. (Have a go at showing this).

(4) Congruence modulo n on $\mathbb{Z}$ for any $n \in \mathbb{N}$. (As the following theorem shows).

Equivalence Relations.

Theorem

Let $n \in \mathbb{N}$ and let $a, b, c \in \mathbb{Z}$. Then

(i) $a \equiv a \pmod{n}$.

(ii) *If* $a \equiv b \pmod{n}$ *then* $b \equiv a \pmod{n}$.

(iii) *If* $a \equiv b \pmod{n}$ *and* $b \equiv c \pmod{n}$*, then* $a \equiv c \pmod{n}$.

- The following proof of transitivity of the congruence modulo n relation is typical.

Proof of (iii).

- Suppose that $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$.
- Then $a - b = kn$ and $b - c = jn$ for some $k, j \in \mathbb{Z}$.
- Adding these two equations we obtain $a - c = (k + j)n$.
- Since $k + j \in \mathbb{Z}$, it follows that $a \equiv c \pmod{n}$. ■

Equivalence Relations.

Definition.

Let A be an non-empty set and let $\sim$ be an equivalence relation on A.

- The relation classes of A with respect to $\sim$ are called equivalence classes.
- The quotient set of A and $\sim$ is the set of all equivalence classes of A with respect to $\sim$, that is, the set $\{[x] \mid x \in A\}$. The quotient set of A and $\sim$ is denoted $A/\sim$. ▲

Example

(1) Let P be the set of all people, and let $\sim$ be the relations on P given by $x \sim y$ iff x and y are the same age (in years).

- If person x is 25 years old, then the equivalence class of x is the set of all 25 year olds.
- The quotient set $P/\sim$ has elements each of which is a set i.e. the set of all one-year olds, the set of all two-year olds, etc.

Equivalence Relations.

(2) Consider the case of congruence modulo n, and let $n = 3$.

- The equivalence classes are

$$\begin{aligned}
&\vdots \\
[0] &= \{\ldots, -6, -3, 0, 3, 6, \ldots\} \\
[1] &= \{\ldots, -5, -2, 1, 4, 7, \ldots\} \\
[2] &= \{\ldots, -4, -1, 2, 5, 8, \ldots\} \\
[3] &= \{\ldots, -3, 0, 3, 6, 9, \ldots\} \\
&\vdots
\end{aligned}$$

  We can see that there are 3 distinct equivalence classes. In general there will be n.

- The quotient set in the general case is called the set of integers modulo n and is the set $\mathbb{Z}_n = \{[0], [1], \ldots, [n-1]\}$. Sometimes it is denoted $\mathbb{Z}/n\mathbb{Z}$.

Equivalence Relations.

(3) Consider the indifference relation $\sim$ on a set of alternatives X.

- For any $x \in X$, the equivalence class $[x]$ is called the indifference class of x and is simply a generalization of the concept of "the indifference curve that passes through x".
- The quotient set in this case is the set of all indifference classes and generalizes the idea of the set of indifference curves.

As part (i) of the following theorem demonstrates, no two distinct indifference sets can have a point in common. This is just the familiar idea that "distinct indifference curves cannot cross".

(4) Equivalence relations are used to construct the integers and rational numbers from the natural numbers. (If interested, see Bloch 8.4).

Equivalence Relations.

Theorem

Let A be a non-empty set, and let $\sim$ be an equivalence relation on A.

(i) *Let $x, y \in A$. If $x \sim y$ then $[x] = [y]$. If $x \not\sim y$, then $[x] \cap [y] = \emptyset$.*

(ii) $\bigcup_{x \in A} [x] = A$.

Example

- In the case of the equivalence relation congruence modulo n, we have $0 \equiv n \pmod{n}$ and so $[0] = [n]$. The distinct equivalence classes are $[0], [1], \ldots, [n-1]$ and these classes are disjoint.
- Part (ii) of the theorem in this case says that $\cdots [-1] \cup [0] \cup [1] \cup [2] \cup \cdots = \mathbb{Z}$. In fact, some of the sets $[x]$ are equal to each other, and for this equivalence relation we can make the stronger statement that $[0] \cup [1] \cup \cdots \cup [n-1] = \mathbb{Z}$.

- The following corollary follows directly from part (i) of the previous theorem.

Equivalence Relations.

Corollary

Let A be a non-empty set, and let $\sim$ be an equivalence relation on A. Let $x, y \in A$. Then $[x] = [y]$ iff $x \sim y$.

- Consider an equivalence relation $\sim$ on a set A.
  - We have seen that any two distinct equivalence classes are disjoint, and that the union of all equivalence classes is the original set A.
  - Thus we can think of the quotient set $A/\sim$ as splitting up A into disjoint subsets. The following definition generalizes this idea.

Definition.

Let A be a non-empty set. A partition of A is a collection $\mathcal{D}$ of non-empty subsets of A such that

(1) $P \cap Q = \emptyset$ when $P, Q \in \mathcal{D}$ and $P \neq Q$;

(2) $\bigcup_{P \in \mathcal{D}} P = A$. ▲

Equivalence Relations.

- Using the above definition, we can now state another corollary to the theorem.

Corollary

Let A be a non-empty set, and let $\sim$ be an equivalence relation on A. Then $A/\sim$ is a partition of A.

Figure: A partition $\mathcal{D} = \{D_1, D_2, D_3, D_4, D_5\}$ of the set A.

Order Relations.

- Next we look at another type of relation. Order relations, as the name suggests, allow us to order the elements of a set in some way.

Definition.

Let A be a non-empty set and let $\precsim$ be a relation on A.

- The relation $\precsim$ is a preorder if it is a transitive and reflexive. The pair $(A, \precsim)$ is called a preordered set.
- The relation $\precsim$ is a partial order if it is an antisymmetric preorder on A. The pair $(A, \precsim)$ is called a partially ordered set also known as a poset.
- The relation $\precsim$ is a linear order if it is a complete partial order on A. The pair $(A, \precsim)$ is called a linearly ordered set also known as a loset. ▲

Order Relations.

Example

(1) Let A be a set. Then $(\mathcal{P}, \subseteq)$ is a poset. The relation $\subseteq$ is not a linear ordering as it is not complete, since for any two subsets $X, Y \subseteq A$, it might not be the case that $X \subseteq Y$ or $Y \subseteq X$.

(2) Each of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ with $\leq$ are linearly ordered sets. The relation $<$ on these sets is not a partial order, since it is not reflexive. Note that, in contrast, the set of complex numbers $\mathbb{C}$ does not have such a naturally occurring linear order.

(3) A rational preference relation is a complete and transitive relation on X and is thus a preorder on X. Note that reflexivity is implied by completeness (prove this). ♦

Order Relations.

- The following definition describes a subset of a partially ordered set with the special property that we can compare any two elements.

Definition.

Let $(A, \precsim)$ be a poset and let $X \subseteq A$. X is a chain in A if $x \precsim y$ or $y \precsim x$ for all $x, y \in X$. ▲

Example

The set of powers of 2, i.e. $\{2^0, 2^1, 2^2, \ldots\}$ is a chain in $\mathbb{N}$ ordered by $|$, the divisibility relation (where we define $m|n$ iff m divides n for all $m, n \in \mathbb{N}$). ♦

- There are different notions of the least and greatest element of a set with respect to some partial ordering. We define them now.

Order Relations.

Definition.

Let $(A, \precsim)$ be a poset and let $X \subseteq A$.

- A greatest element or maximum of X is an element $p \in X$ such that $x \precsim p$ for all $x \in X$.
- A maximal element of X is an element $p \in X$ such that there exists no $x \in X$ with $p \prec x$.
- A least element or minimum of X is an element $q \in X$ such that $q \precsim x$ for all $x \in X$.
- A minimal element of X is an element $q \in X$ such that there exists no $x \in X$ with $x \prec q$. ▲

Example

Consider the set $\mathbb{N}$ ordered by the divisibility relation $|$. This set has least element 1 in the ordering $|$, but no greatest element. Let $B = \{2, 3, 4, \ldots\}$. The subset B has no least element ($2|3$ fails), has infinitely many minimal elements (2,3,5, etc – the prime numbers). B has no greatest or maximal elements. ⧫

Order Relations.

Theorem

Let $(A, \precsim)$ be a poset and let $X \subseteq A$.

(i) *If X has a greatest element, then it is unique.*

(ii) *If X has a greatest element, then it is maximal.*

(iii) *If X is a chain, then every maximal element of X is a greatest element.*

(iv) *If X has a least element, then it is unique.*

(v) *If X has a least element, then it is minimal.*

(vi) *If X is a chain, then every minimal element of X is a least element.*

- If X is a chain we use $\max X$ to denote the maximal and greatest element, and $\min X$ to denote the minimal and least element (if they exist).

Order Relations.

Definition.

Let $(A, \precsim)$ be a poset and let $X \subseteq A$.

- An upper bound for X is an element $p \in A$ such that $x \precsim p$ for all $x \in X$.
- A least upper bound or supremum for X is an element $p \in A$ such that p is an upper bound for X, and such that $p \precsim z$ for any other upper bound z for X. This is denoted by $\sup X$.
- A lower bound for X is an element $q \in A$ such that $q \precsim x$ for all $x \in X$.
- A greatest lower bound or infimum for X is an element $q \in A$ such that q is an lower bound for X, and such that $w \precsim q$ for any other lower bound w for X. This is denoted by $\inf X$. ▲

Order Relations.

Example

(1) Consider the poset $(\mathbb{Q}, \leq)$.

- The set $X = \{1/2, 2/2, 3/2, 4/2, 5/2/\ldots\}$ has no upper bound in $\mathbb{Q}$, and hence no supremum. This set does have many lower bounds, for example -999 and 0, and it has a greatest lower bound $1/2$.
- Now consider $Y = \{x \in \mathbb{Q} \mid 1 < x < 3\}$. The set Y has many upper and lower bounds. It's supremum is 3, and its infimum is 1. Unlike the set X, which contains its infimum, the set Y contains neither its infimum nor its supremum.
- Next consider $Z = \{x \in \mathbb{Q} \mid 0 \leq x < \sqrt{2}\}$. The set Z has an infimum of 0. However, although Z has many upper bounds in $\mathbb{Q}$, such as 2 and $3/2$, it has no supremum in $\mathbb{Q}$, since $\sqrt{2} \notin \mathbb{Q}$.

Order Relations.

In contrast, if we look at the poset $(\mathbb{R}, \leq)$, the problem discussed above does not occur. If we let $Z' = \{x \in \mathbb{R} \mid 0 \leq x < \sqrt{2}\}$, then this set has a supremum in $\mathbb{R}$, namely $\sqrt{2}$. In fact, the property which distinguishes $\mathbb{R}$ from $\mathbb{Q}$ is that every set in $\mathbb{R}$ that has an upper bound must have a least upper bound.

(2) For any non-empty set, the poset $(\mathcal{P}(A), \subseteq)$ always has a supremum and infimum. Let $X \subseteq \mathcal{P}(A)$. Note X is a collection of subsets of A. The supremum for X is $\bigcup_{D \in X} D$ and the infimum for X is $\bigcap_{D \in X} D$. We saw this in our theorem for an indexed family of sets. ♦

Order Relations.

Theorem

Let $(A, \precsim)$ be a poset and let $X \subseteq A$.

(i) *If X has a supremum, then it is unique.*

(ii) *If X has an infimum, then it is unique.*

Proof.

Suppose that $p, q \in A$ are both suprema for X. Then, by definition, both p and q are upper bounds for X.

- Since p is a supremum for X, and q is an upper bound for X, then $p \precsim q$ by the definition of suprema.
- Also since q is a supremum, we see that $q \precsim p$.

By antisymmetry (remember $\precsim$ is a partial order and so it is an antisymmetric relation) it follows that $p = q$. ■

- The final theorem describes when a supremum is a greatest element and when an infimum is a least element.

Order Relations.

Theorem

Let $(A, \precsim)$ be a poset and let $X \subseteq A$.

(i) *If p is the supremum of X and $p \in X$, then p is the greatest element in X.*

(ii) *If q is the infimum of X and $q \in X$, then q is the least element in X.*

Example

Consider the linear order $\leq$ on $\mathbb{R}$ and the subsets $B_1 = (0, 1)$ $B_2 = [0, 1)$ $B_3 = (0, \infty)$ $B_4 = (-\infty, 0)$.

- B_1 has no least element and no greatest. Any $b \leq 0$ is a lower bound, so $\inf B_1 = 0$, similarly $\sup B_1 = 1$.
- B_2 has a least element $\min B_2 = \inf B_2 = 0$, and again $\sup B_2 = 1$, but no greatest element.
- B_3 has no greatest element or supremum and indeed no upper bound. $\inf B_3 = 0$.
- Similarly for B_4. ♦