

Relations

Relations.

Definition. Let A and B be sets.

- A relation R from A to B is a subset $R \subseteq A \times B$.
- If $a \in A$ and $b \in B$, we write $a R b$ if $(a, b) \in R$, and $a \not R b$ if $(a, b) \notin R$.
- A relation from A to A is called a *relation on A* . ▲

- So, if $(x, y) \in R$ or $(y, x) \in R$, we then think of R as associating the object x with y . If $\{(x, y), (y, x)\} \cap R = \emptyset$ then there is no connection between x and y embodied in R .

Example 1.

1. Let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$. There are many possible relations from A to B , one example of which is the relation S defined by $S = \{(1, y), (1, z), (2, y)\}$. Then $1 S y$, $1 S z$, and $2 S y$.
2. The symbols $<$ and \leq both represent relations on \mathbb{R} .
3. Let $P = \{\text{♣}, \text{♠}, \text{♥}\}$ and let H be the relation on P defined by having an object x related to an object y iff x is heavier than y . Then $H = \{(\text{♣}, \text{♠}), (\text{♣}, \text{♥}), (\text{♠}, \text{♥})\}$ or $\text{♣} H \text{♠}$, $\text{♣} H \text{♥}$, and $\text{♠} H \text{♥}$.
4. Let A be a set. Define a relation on $\mathcal{P}(A)$ by saying that $P, Q \in \mathcal{P}(A)$ are related iff $P \subseteq Q$.
5. In economics, the preference-based approach summarizes the objectives of a decision maker in a *preference relation* on the set of alternatives X . This relation is denoted by \succsim and, for any $x, y \in X$, we read $x \succsim y$ as “ x is at least as good as y ”. ◆

Definition. Let A and B be non-empty sets, and let R be a relation from A to B . For each $x \in A$, define the *relation class* of x with respect to R , denoted $R[x]$, to be the set

$$R[x] = \{y \in B \mid x R y\}$$

We will sometimes write $[x]$ instead of $R[x]$ if the relation R is understood from the context. ▲

Example 2. We continue examples (1) and (2) from above.

1. For this relation we have $[1] = \{y, z\}$, we have $[2] = \{y\}$ and we have $[3] = \emptyset$.
2. For the relation $<$, we have $[x] = (x, \infty)$ for all $x \in \mathbb{R}$, and for the relation \leq , we have $[x] = [x, \infty)$ for all $x \in \mathbb{R}$. ◆

Definition. Let A be a non-empty set and let R be a relation on A .

- The relation R is *complete* if either $x R y$ or $y R x$ holds for all $x, y \in A$.
- The relation R is *reflexive* if $x R x$ for all $x \in A$.
- The relation R is *irreflexive* if $x \not R x$ for all $x \in A$.
- The relation R is *symmetric* if $x R y$ implies $y R x$ for all $x, y \in A$.
- The relation R is *asymmetric* if $x R y$ implies $y \not R x$ for all $x, y \in A$.
- The relation R is *antisymmetric* if $x R y$ and $y R x$ imply $x = y$ for all $x, y \in A$.
- The relation R is *transitive* if $x R y$ and $y R z$ imply $x R z$ for all $x, y, z \in A$. ▲

Example 3.

1. Let $B = \{x, y, z\}$, and let $T = \{(x, x), (y, y), (z, z), (x, y), (y, z)\}$. Then T is reflexive, but neither symmetric nor transitive. It is reflexive because (x, x) , (y, y) and (z, z) are all in T . The relation is not symmetric, since $x T y$ but $y \not T x$. The relation is not transitive, since $x T y$ and $y T z$ but $x \not T z$.
 2. The relation $<$ on \mathbb{R} is transitive, but is not symmetric or reflexive. Let $x, y, z \in \mathbb{R}$. If $x < y$ and $y < z$, then $x < z$, and so the relation is transitive. The relation is not symmetric since if $x < y$, it is never true that $y < x$. It is not reflexive because we never have $x < x$.
 3. The relation \leq on \mathbb{R} is complete, reflexive (so not irreflexive), transitive and antisymmetric (but not symmetric).
 4. A preference relation \succsim is a *rational preference relation* if it is both transitive and complete. ◆
- We can also create new relations from old.

Definition. Let A be a set and let R be a reflexive relation on A .

- The *asymmetric part* of R is the relation P_R on A , defined by $x P_R y$ iff $x R y$ and $y \not R x$.
- The *symmetric part* of R is the relation I_R on A , defined by $x I_R y$ iff $x R y$ and $y R x$. ▲

Example 4. For the preference relation \succsim on a set of alternatives X , the asymmetric part is denoted \succ and $x \succ y$ is read “ x is preferred to y ”. The symmetric part is denoted \sim and $x \sim y$ is read “ x is indifferent to y ”. ◆

Example 5. Another important type of relation on the integers is related to the idea of “clock arithmetic”.

- Suppose it is 2 o'clock, and we want to know what time it will be in 3 hours. Clearly it will be $2 + 3 = 5$ o'clock.
- Now suppose it is 7 o'clock, and you want the time it will be in 7 hours. Proceeding as above would give $7 + 7 = 14$ o'clock, but the correct answer is obviously 2 o'clock, which we get by subtracting 12 from 14, because 14 is greater than 12.

We can generalize this idea.

Definition. Let $n \in \mathbb{N}$. If $a, b \in \mathbb{Z}$, we say that a is *congruent to b modulo n* , denoted $a \equiv b \pmod{n}$ if $a - b = kn$ for some $k \in \mathbb{Z}$. ▲

- For each fixed $n \in \mathbb{N}$, we obtain a relation on \mathbb{Z} given by congruence modulo n . ◆

Equivalence Relations.

Definition. Let A be a set and let \sim be a relation on A . The relation \sim is an *equivalence relation* if it is reflexive, symmetric and transitive. ▲

Example 6. The following are equivalence relations.

1. Equality on the set \mathbb{R} .
2. Being the same age on the set of all people.
3. For rational preferences, the indifference relation \sim . (Have a go at showing this).
4. Congruence modulo n on \mathbb{Z} for any $n \in \mathbb{N}$. (As the following theorem shows).

Theorem 1. Let $n \in \mathbb{N}$ and let $a, b, c \in \mathbb{Z}$. Then

1. $a \equiv a \pmod{n}$.
2. If $a \equiv b \pmod{n}$ then $b \equiv a \pmod{n}$.
3. If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$.

- The following proof of transitivity of the congruence modulo n relation is typical.

Proof of (3).

- Suppose that $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$.
- Then $a - b = kn$ and $b - c = jn$ for some $k, j \in \mathbb{Z}$.
- Adding these two equations we obtain $a - c = (k + j)n$.

- Since $k + j \in \mathbb{Z}$, it follows that $a \equiv c \pmod{n}$. ■

Definition. Let A be a non-empty set and let \sim be an equivalence relation on A .

- The relation classes of A with respect to \sim are called *equivalence classes*.
- The *quotient set* of A and \sim is the set of all equivalence classes of A with respect to \sim , that is, the set $\{[x] \mid x \in A\}$. The quotient set of A and \sim is denoted A/\sim . ▲

Example 7.

1. Let P be the set of all people, and let \sim be the relations on P given by $x \sim y$ iff x and y are the same age (in years).
 - If person x is 25 years old, then the equivalence class of x is the set of all 25 year olds.
 - The quotient set P/\sim has elements each of which is a set i.e. the set of all one-year olds, the set of all two-year olds, etc.

2. Consider the case of congruence modulo n , and let $n = 3$.

- The equivalence classes are

$$\begin{aligned}
 & \vdots \\
 [0] &= \{\dots, -6, -3, 0, 3, 6, \dots\} \\
 [1] &= \{\dots, -5, -2, 1, 4, 7, \dots\} \\
 [2] &= \{\dots, -4, -1, 2, 5, 8, \dots\} \\
 [3] &= \{\dots, -3, 0, 3, 6, 9, \dots\} \\
 & \vdots
 \end{aligned}$$

We can see that there are 3 distinct equivalence classes. In general there will be n .

- The quotient set in the general case is called the set of *integers modulo n* and is the set $\mathbb{Z}_n = \{[0], [1], \dots, [n - 1]\}$. Sometimes it is denoted $\mathbb{Z}/n\mathbb{Z}$.

3. Consider the indifference relation \sim on a set of alternatives X .

- For any $x \in X$, the equivalence class $[x]$ is called the *indifference class* of x and is simply a generalization of the concept of “the indifference curve that passes through x ”.

- The quotient set in this case is the set of all indifference classes and generalizes the idea of the set of indifference curves.

As part (1) of the following theorem demonstrates, no two distinct indifference sets can have a point in common. This is just the familiar idea that “distinct indifference curves cannot cross”.

4. Equivalence relations are used to construct the integers and rational numbers from the natural numbers. (If interested, see Bloch 8.4).

Theorem 2. *Let A be a non-empty set, and let \sim be an equivalence relation on A .*

1. *Let $x, y \in A$. If $x \sim y$ then $[x] = [y]$. If $x \not\sim y$, then $[x] \cap [y] = \emptyset$.*
2. $\bigcup_{x \in A} [x] = A$.

Example 8.

1.
 - In the case of the equivalence relation congruence modulo n , we have $0 \equiv n \pmod{n}$ and so $[0] = [n]$. The distinct equivalence classes are $[0], [1], \dots, [n-1]$ and these classes are disjoint.
 - Part (2) of the theorem in this case says that $\dots [-1] \cup [0] \cup [1] \cup [2] \cup \dots = \mathbb{Z}$. In fact, some of the sets $[x]$ are equal to each other, and for this equivalence relation we can make the stronger statement that $[0] \cup [1] \cup \dots \cup [n-1] = \mathbb{Z}$.
- The following corollary follows directly from part (i) of the previous theorem.

Corollary 1. *Let A be a non-empty set, and let \sim be an equivalence relation on A . Let $x, y \in A$. Then $[x] = [y]$ iff $x \sim y$.*

- Consider an equivalence relation \sim on a set A .
 - We have seen that any two distinct equivalence classes are disjoint, and that the union of all equivalence classes is the original set A .
 - Thus we can think of the quotient set A/\sim as splitting up A into disjoint subsets. The following definition generalizes this idea.

Definition. Let A be a non-empty set. A *partition* of A is a collection \mathcal{D} of non-empty subsets of A such that

1. $P \cap Q = \emptyset$ when $P, Q \in \mathcal{D}$ and $P \neq Q$;
2. $\bigcup_{P \in \mathcal{D}} P = A$. ▲

- Using the above definition, we can now state another corollary to the theorem.

Corollary 2. *Let A be a non-empty set, and let \sim be an equivalence relation on A . Then A/\sim is a partition of A .*

Figure 1: A partition $\mathcal{D} = \{D_1, D_2, D_3, D_4, D_5\}$ of the set A .

Order Relations.

- Next we look at another type of relation. Order relations, as the name suggests, allow us to order the elements of a set in some way.

Definition. Let A be a non-empty set and let \succsim be a relation on A .

- The relation \succsim is a *preorder* if it is a transitive and reflexive. The pair (A, \succsim) is called a *preordered set*.
- The relation \succsim is a *partial order* if it is an antisymmetric preorder on A . The pair (A, \succsim) is called a *partially ordered set* also known as a *poset*.
- The relation \succsim is a *linear order* if it is a complete partial order on A . The pair (A, \succsim) is called a *linearly ordered set* also known as a *loset*. ▲

Example 9.

1. Let A be a set. Then (\mathcal{P}, \subseteq) is a poset. The relation \subseteq is not a linear ordering as it is not complete, since for any two subsets $X, Y \subseteq A$, it might not be the case that $X \subseteq Y$ or $Y \subseteq X$.
 2. Each of the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and \mathbb{R} with \leq are linearly ordered sets. The relation $<$ on these sets is not a partial order, since it is not reflexive. Note that, in contrast, the set of complex numbers \mathbb{C} does not have such a naturally occurring linear order.
 3. A rational preference relation is a complete and transitive relation on X and is thus a preorder on X . Note that reflexivity is implied by completeness (prove this). ◆
- The following definition describes a subset of a partially ordered set with the special property that we can compare any two elements.

Definition. Let (A, \succsim) be a poset and let $X \subseteq A$. X is a chain in A if $x \succsim y$ or $y \succsim x$ for all $x, y \in X$. ▲

Example 10. The set of powers of 2, i.e. $\{2^0, 2^1, 2^2, \dots\}$ is a chain in \mathbb{N} ordered by $|$, the divisibility relation (where we define $m|n$ iff m divides n for all $m, n \in \mathbb{N}$). ◆

- There are different notions of the least and greatest element of a set with respect to some partial ordering. We define them now.

Definition. Let (A, \succsim) be a poset and let $X \subseteq A$.

- A *greatest element* or *maximum* of X is an element $p \in X$ such that $x \succsim p$ for all $x \in X$.

- A *maximal element* of X is an element $p \in X$ such that there exists no $x \in X$ with $p \prec x$.
- A *least element* or *minimum* of X is an element $q \in X$ such that $q \preceq x$ for all $x \in X$.
- A *minimal element* of X is an element $q \in X$ such that there exists no $x \in X$ with $x \prec q$. ▲

Example 11. Consider the set \mathbb{N} ordered by the divisibility relation $|$. This set has least element 1 in the ordering $|$, but no greatest element. Let $B = \{2, 3, 4, \dots\}$. The subset B has no least element ($2|3$ fails), has infinitely many minimal elements (2,3,5, etc – the prime numbers). B has no greatest or maximal elements. ◆

Theorem 3. Let (A, \preceq) be a poset and let $X \subseteq A$.

1. If X has a greatest element, then it is unique.
 2. If X has a greatest element, then it is maximal.
 3. If X is a chain, then every maximal element of X is a greatest element.
 4. If X has a least element, then it is unique.
 5. If X has a least element, then it is minimal.
 6. If X is a chain, then every minimal element of X is a least element.
- If X is a chain we use $\max X$ to denote the maximal and greatest element, and $\min X$ to denote the minimal and least element (if they exist).

Definition. Let (A, \preceq) be a poset and let $X \subseteq A$.

- An *upper bound* for X is an element $p \in A$ such that $x \preceq p$ for all $x \in X$.
- A *least upper bound* or *supremum* for X is an element $p \in A$ such that p is an upper bound for X , and such that $p \preceq z$ for any other upper bound z for X . This is denoted by $\sup X$.
- A *lower bound* for X is an element $q \in A$ such that $q \preceq x$ for all $x \in X$.
- A *greatest lower bound* or *infimum* for X is an element $q \in A$ such that q is a lower bound for X , and such that $w \preceq q$ for any other lower bound w for X . This is denoted by $\inf X$. ▲

Example 12.

1. Consider the poset (\mathbb{Q}, \leq) .
 - The set $X = \{1/2, 2/2, 3/2, 4/2, 5/2/ \dots\}$ has no upper bound in \mathbb{Q} , and hence no supremum. This set does have many lower bounds, for example -999 and 0 , and it has a greatest lower bound $1/2$.

- Now consider $Y = \{x \in \mathbb{Q} \mid 1 < x < 3\}$. The set Y has many upper and lower bounds. Its supremum is 3, and its infimum is 1. Unlike the set X , which contains its infimum, the set Y contains neither its infimum nor its supremum.
- Next consider $Z = \{x \in \mathbb{Q} \mid 0 \leq x < \sqrt{2}\}$. The set Z has an infimum of 0. However, although Z has many upper bounds in \mathbb{Q} , such as 2 and $3/2$, it has no supremum in \mathbb{Q} , since $\sqrt{2} \notin \mathbb{Q}$.

1. In contrast, if we look at the poset (\mathbb{R}, \leq) , the problem discussed above does not occur. If we let $Z' = \{x \in \mathbb{R} \mid 0 \leq x < \sqrt{2}\}$, then this set has a supremum in \mathbb{R} , namely $\sqrt{2}$. In fact, the property which distinguishes \mathbb{R} from \mathbb{Q} is that every set in \mathbb{R} that has an upper bound must have a least upper bound.
2. For any non-empty set, the poset $(\mathcal{P}(A), \subseteq)$ always has a supremum and infimum. Let $X \subseteq \mathcal{P}(A)$. Note X is a collection of subsets of A . The supremum for X is $\bigcup_{D \in X} D$ and the infimum for X is $\bigcap_{D \in X} D$. We saw this in our theorem for an indexed family of sets. \blacklozenge

Theorem 4. Let (A, \lesssim) be a poset and let $X \subseteq A$.

1. If X has a supremum, then it is unique.
2. If X has an infimum, then it is unique.

Proof. Suppose that $p, q \in A$ are both suprema for X . Then, by definition, both p and q are upper bounds for X .

- Since p is a supremum for X , and q is an upper bound for X , then $p \lesssim q$ by the definition of suprema.
- Also since q is a supremum, we see that $q \lesssim p$.

By antisymmetry (remember \lesssim is a partial order and so it is an antisymmetric relation) it follows that $p = q$. \blacksquare

- The final theorem describes when a supremum is a greatest element and when an infimum is a least element.

Theorem 5. Let (A, \lesssim) be a poset and let $X \subseteq A$.

1. If p is the supremum of X and $p \in X$, then p is the greatest element in X .
2. If q is the infimum of X and $q \in X$, then q is the least element in X .

Example 13. Consider the linear order \leq on \mathbb{R} and the subsets $B_1 = (0, 1)$ $B_2 = [0, 1)$ $B_3 = (0, \infty)$ $B_4 = (-\infty, 0)$.

- B_1 has no least element and no greatest. Any $b \leq 0$ is a lower bound, so $\inf B_1 = 0$, similarly $\sup B_1 = 1$.

- B_2 has a least element $\min B_2 = \inf B_2 = 0$, and again $\sup B_2 = 1$, but no greatest element.
- B_3 has no greatest element or supremum and indeed no upper bound. $\inf B_3 = 0$.
- Similarly for B_4 . ♦