

MTAEA – Metric Space Properties of Euclidean Spaces

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February 8, 2010

Sequences in \mathbb{R}^m .

Definition.

A **sequence** s in \mathbb{R}^m is any function $s : \mathbb{N} \rightarrow \mathbb{R}^m$. ▲

- ▶ A sequence in \mathbb{R}^m is an assignment of a vector in \mathbb{R}^m to each natural number.
- ▶ For such sequences there are two indices to keep track of:
 - ▶ one for the m coordinates of each m -vector, and
 - ▶ the other to indicate the element in the sequence.
- ▶ We say a sequence in \mathbb{R}^m converges, if as n gets large the terms get “close” to some number s .
- ▶ Before we define convergence, we need a way to measure the “closeness” of elements in the sequence.

Sequences in \mathbb{R}^m .

- ▶ We use the notion of distance between vectors we learnt earlier.
- ▶ The distance between any two vectors $x, y \in \mathbb{R}^m$ is the norm of their difference:

$$\|x - y\| = \sqrt{(x - y) \cdot (x - y)} = \sqrt{(x_1 - y_1)^2 + \cdots + (x_m - y_m)^2}.$$

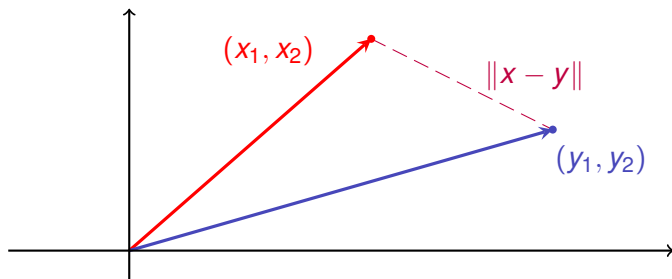


Figure: The distance between two vectors in the plane.

Sequences in \mathbb{R}^m .

Definition.

The standard **Euclidean metric** on \mathbb{R}^m is given by

$$d(x, y) = \|x - y\| = \left[\sum_{k=1}^m (x_k - y_k)^2 \right]^{\frac{1}{2}}$$



- ▶ As we saw in our study of Euclidean vector spaces, the Euclidean metric satisfies the following properties.
 - (1) $d(x, x) = 0$ for all $x \in \mathbb{R}^m$ and $d(x, y) > 0$ $x \neq y \in \mathbb{R}^m$.
 - (2) $d(x, y) = d(y, x)$ for all $x, y \in \mathbb{R}^m$.
 - (3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \mathbb{R}^m$.
- ▶ In fact any set S together with a function d satisfying the above properties – called a **distance function** or **metric** – is referred to as a **metric space**.

Sequences in \mathbb{R}^m .

- ▶ The following definition generalizes to \mathbb{R}^m the idea of the ε interval about a point a given by $I_\varepsilon(a) = \{x \in \mathbb{R} \mid |x - a| < \varepsilon\}$.

Definition.

Let $\varepsilon > 0$ and let a be a vector in \mathbb{R}^m . The **open ε -ball about a** is

$$B_\varepsilon(a) = \{x \in \mathbb{R}^m \mid \|x - a\| < \varepsilon\}.$$

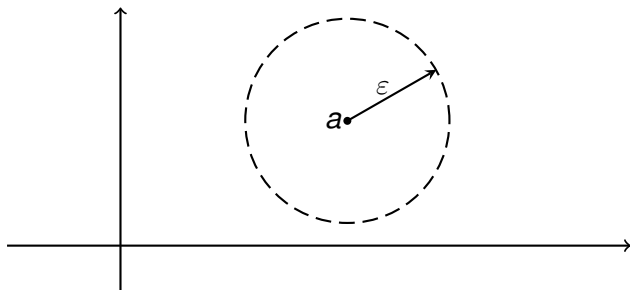


Figure: The open ε -ball about $a \in \mathbb{R}^2$.

Sequences in \mathbb{R}^m .

- ▶ So all we have to do is $|\cdot|$ with $\|\cdot\|$ in our definition of convergence.

Definition.

A sequence (s_n) in a Euclidean space \mathbb{R}^m **converges** to s if

for each $\varepsilon > 0$ there exists a number N , such that $n > N$ implies that $s_n \in B_\varepsilon(s)$.

The vector s is called the **limit** of (s_n) . ▲

- ▶ Thus the sequence of vectors $(s_n)_{n \in \mathbb{N}}$ converges to a limit vector s if the sequence of distances between the vectors s_n and s , given by $(\|s_n - s\|)_{n \in \mathbb{N}}$ converges to 0.
- ▶ Equivalently, we say a sequence (s_n) converges to s if $\lim_{n \rightarrow +\infty} d(s_n, s) = 0$. (This definition is more general, and applies to any metric space.)

Sequences in \mathbb{R}^m .

Example

Consider the sequence (s_n) in \mathbb{R}^2 given by

$$\left(\left(1, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{3}\right), \left(-\frac{1}{3}, -\frac{1}{4}\right), \left(-\frac{1}{4}, \frac{1}{5}\right), \left(\frac{1}{5}, \frac{1}{6}\right), \dots \right)$$

The terms in the sequence move clockwise in \mathbb{R}^2 , as they converge to the origin. Notice that each term s_n gets closer to the origin, each of its components is also getting closer to the origin. ◆

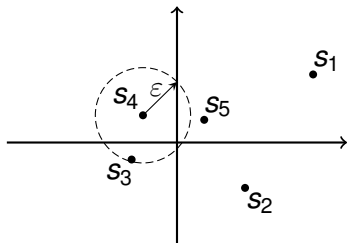


Figure: The sequence (s_n) in \mathbb{R}^2 converging to 0. For the given ϵ we need $n > 3$.

Sequences in \mathbb{R}^m .

- ▶ The findings of the last example are true of any convergent sequence in \mathbb{R}^m as the following theorem states.

Theorem (S&B 12.5)

A sequence (s_n) in \mathbb{R}^m converges iff all of its m component sequences converge in \mathbb{R} .

Proof.

We will prove the “only if” part. Let $s_n = (s_{1n}, \dots, s_{mn})$. Suppose that (s_n) converges to \hat{s} and let $\varepsilon > 0$. Then there exists an N such that $n > N$ implies $s_n \in B_\varepsilon(\hat{s})$ or, equivalently, $\|s_n - \hat{s}\| < \varepsilon$. We want to show any component sequence $(s_{kn})_{n \in \mathbb{N}}$ of $(s_n)_{n \in \mathbb{N}}$ converges to \hat{s}_k . Now, for $n > N$ we have

$$\begin{aligned} |s_{kn} - \hat{s}_k| &= \sqrt{(s_{kn} - \hat{s}_k)^2} \leq \sqrt{(s_{1n} - \hat{s}_1)^2 + \dots + (s_{mn} - \hat{s}_m)^2} \\ &= \|s_n - \hat{s}\| < \varepsilon. \end{aligned}$$



Sequences in \mathbb{R}^m .

- ▶ This result reduces the problem of proving convergence of a sequence of vectors in \mathbb{R}^m to the problem of proving the convergence of m sequences in \mathbb{R} .
- ▶ This allows us to apply results about sequences in \mathbb{R} to sequences in \mathbb{R}^m .
 - ▶ From the previous theorem and uniqueness of limits of sequences in \mathbb{R} it follows that the limit of a sequence in \mathbb{R}^m is unique.
- ▶ The next result generalizes our limit theorem for convergent sequences in \mathbb{R} .

Theorem (S&B 12.6)

Suppose that (s_n) and (t_n) are convergent sequences of vectors in \mathbb{R}^m , with limits s and t respectively; and let (k_n) be a convergent sequence of real numbers with limit k .

- (i) $\lim(k_n s_n) = ks$.
- (ii) $\lim(s_n + t_n) = s + t$.
- (iii) $\lim(s_n \cdot t_n) = s \cdot t$.

Open Sets.

- ▶ The next definition formalizes the idea that a set is “open” if it has “no boundary”.

Definition.

Let S be a subset of \mathbb{R}^m .

- ▶ We say S is **open** in \mathbb{R}^m if, for each $x \in S$, there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq S$. ▲
- ▶ So the set S is open if there exists an open ε -ball about x completely contained in S .
- ▶ This means we can always move a small distance in any direction from any point in the set and still be strictly inside the set.

Open Sets.

Example

- ▶ The set \mathbb{R}^m is open in \mathbb{R}^m .
- ▶ Any open interval (a, b) in \mathbb{R} is open!
- ▶ Any half open interval $(a, b]$ or $[b, a)$ in \mathbb{R} is not an open set.
- ▶ Any point in any Euclidean space is not an open set.
- ▶ Any line in \mathbb{R}^2 , any plane in \mathbb{R}^3 , or any hyperplane in \mathbb{R}^m is not open.
- ▶ Any object of lower dimension than the Euclidean space it is in is not open. ♦

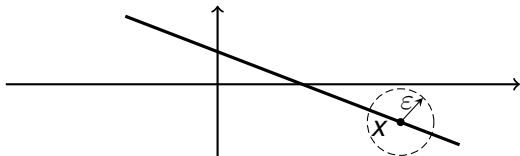


Figure: A disc about any point x on a line in \mathbb{R}^2 contains points not on the line.

Open Sets.

Example

Show the interval $(0, 1)$ is an open set.

Proof.

Take any point $x \in (0, 1)$. Then $0 < x < 1$ and so

$$0 < \frac{x}{2} = x - \frac{x}{2} < x < 1$$

and

$$0 < x < x + \frac{1-x}{2} = \frac{1+x}{2} < 1$$

So let $\varepsilon = \min\{x/2, (1-x)/2\}$. Then the open ε -interval about x given by $(x - \varepsilon, x + \varepsilon)$ is strictly contained in $(0, 1)$. ■

Open Sets.

Theorem (S&B 12.7)

Open balls are open sets.

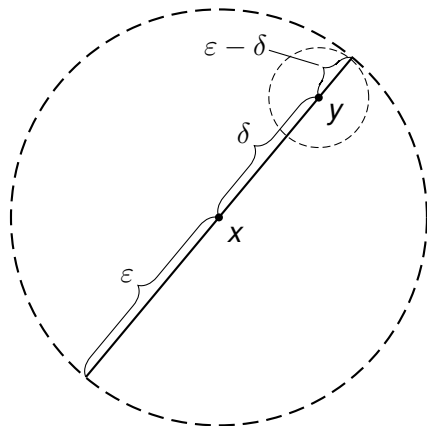


Figure: If $y \in B_\epsilon(x)$ and $\delta = \|x - y\|$, then $B_{\epsilon-\delta}(y) \subseteq B_\epsilon(x)$.

Open Sets.

Theorem (S&B 12.8)

- (i) *The union of any collection of open sets is open.*
- (ii) *The intersection of **finitely many** open sets is open.*

Proof of (ii).

Let S_1, \dots, S_n be open sets, and let $S = \bigcap_{i=1}^n S_i$. Let $x \in S$.


- ▶ To show that S is open, we need to show that there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq S$.
- ▶ Now, since $x \in S$ we have $x \in S_i$ for $i = 1, \dots, n$.
- ▶ Since each S_i is open, for each i there exists an ε_i such that $B_{\varepsilon_i}(x) \subseteq S_i$.
- ▶ Let $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$. The ball $B_\varepsilon(x)$ is contained in each $B_{\varepsilon_i}(x)$ and so $B_\varepsilon(x) \subseteq S_i$ for all $i = 1, \dots, n$.

Thus $B_\varepsilon(x) \subseteq S$ and so S is open. ■

Open Sets.

Example

To see why an intersection of infinitely many open sets need not be open, consider the open intervals $S_n = (-1/n, 1/n)$ in \mathbb{R} for $n \in \mathbb{N}$.

- ▶ A set S_n is simply an interval of radius $1/n$ about 0.
- ▶ The intersection of these sets is $\bigcap_{n \in \mathbb{N}} S_n = \{0\}$ since if $y \neq 0$, $y \notin S_n$ for all $n > 1/|y|$.
- ▶ But a point can never be open, so the infinite intersection of the sets S_n is not open. 

Open Sets.

Definition.

Let S be a subset of \mathbb{R}^m .

- ▶ A point $x \in \mathbb{R}^m$ is an **interior point** of S if, for some $\varepsilon > 0$, $B_\varepsilon(x) \subseteq S$.
- ▶ The **interior** of S , denoted by $\text{int } S$ is the set of all interior points of S . ▲
- ▶ So x is an interior point of S if there is some open ball about x contained in S .

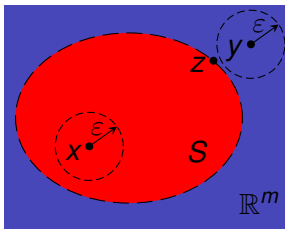


Figure: Point x is an interior point of the open set S , but points y and z are not.

Open Sets.

- ▶ The interior of S is the union of all open sets contained in S – this is sometimes used as the definition of the interior of S .
- ▶ By the previous theorem, the interior is an open set.
- ▶ The interior of S is the largest open set contained in S .
- ▶ Thus a set S is open in \mathbb{R}^m iff every point in S is interior to S , i.e. iff $S = \text{int } S$.

Example

- ▶ Consider the interval $[a, b]$ in \mathbb{R} .
- ▶ A point x is interior to $[a, b]$ if for some $\varepsilon > 0$, the set $\{y \in \mathbb{R} \mid |x - y| < \varepsilon\}$ is a subset of $[a, b]$.
- ▶ That is, x is interior if for all y such that $x - \varepsilon < y < x + \varepsilon$, we have $a \leq y \leq b$.
- ▶ Thus we need $a \leq x - \varepsilon$ and $x + \varepsilon \leq b$ for some $\varepsilon > 0$.
- ▶ Hence a point x is interior if $a < x < b$ i.e. if $x \in (a, b)$. ◆

Closed Sets.

- ▶ The next definition formalizes the idea that set is closed if it contains all its “boundary points.”

Definition.

Let S be a subset of \mathbb{R}^m . We say S is **closed** in \mathbb{R}^m if, whenever (s_n) is a convergent sequence such that $s_n \in S$ for all $n \in \mathbb{N}$, its limit is also contained in S . ▲

- ▶ Thus a set is closed if it contains all its limit points.
- ▶ This means that for any given point x , if there are points in the set arbitrarily close to x , then x must be in the set too.
- ▶ Closedness and openness are complementary properties – a closed set contains its boundary points, an open set does not.
- ▶ We will use this property to provide another definition of a closed set. But first we define the complement of a set.

Closed Sets.

Definition.

Let S be a subset of \mathbb{R}^m . The **complement** of S , denoted S^c set defined by

$$S^c = \{x \in \mathbb{R}^m \mid x \notin S\} = \mathbb{R}^m \setminus S. \quad \blacktriangle$$

- ▶ Thus the complement of S is the set of all points that are not in S .
- ▶ We now state the result that the complement of an open set is closed and the complement of a closed set is open.

Theorem (S&B 12.9)

A subset S of \mathbb{R}^m is closed iff if its complement S^c is open.

- ▶ This theorem provides an alternative definition of a closed set as a set whose complement is open.

Closed Sets.

Example

- ▶ Any set of the form $(-\infty, a] \cup [b, +\infty)$ is closed, since this set is the complement of the set (a, b) .
- ▶ The set \mathbb{R}^m is closed in \mathbb{R}^m because any convergent sequence in \mathbb{R}^m has a limit in \mathbb{R}^m . Note that \mathbb{R}^m is both closed and open.
- ▶ The empty set is also closed and open in \mathbb{R}^m . It is the complement of \mathbb{R}^m . ◆

Theorem (S&B 12.10)

- (i) *The intersection of any collection of closed sets is closed.*
- (ii) *The union of **finitely many** closed sets is closed.*

Proof.

Use theorem (5), theorem (4) and the fact that $(\bigcup_i S_i)^c = \bigcap_i (S_i^c)$. ■

Closed Sets.

Example

To see why a union of infinitely many closed sets need not be closed, consider the closed intervals $S_n = [-n/(n+1), n/(n+1)]$ in \mathbb{R} for $n \in \mathbb{N}$.

- ▶ The set S_n is the interval of radius $n/(n+1)$ about 0.
- ▶ The union of these sets is $\bigcup_{n \in \mathbb{N}} S_n = (-1, 1)$.
- ▶ To show this, consider any $y \in \bigcup_{n \in \mathbb{N}} S_n$. Then $y \in S_n$ for some $n \in \mathbb{N}$ and so $-n/(n+1) \leq y \leq n/(n+1)$ for some n .
- ▶ It follows that $y \in (-1, 1)$ and that $\bigcup_{n \in \mathbb{N}} S_n \subseteq (-1, 1)$.
- ▶ Now, let $y \in (-1, 1)$. Then $-1 < y < 1$, and so we can find an $n \in \mathbb{N}$ such that $-n/(n+1) \leq y \leq n/(n+1)$.
- ▶ Thus $(-1, 1) \subseteq \bigcup_{n \in \mathbb{N}} S_n$.
- ▶ But the set $(-1, 1)$ is open. ◆

Closed Sets.

Definition.

Let S be a subset of \mathbb{R}^m .

- ▶ A point $x \in \mathbb{R}^m$ is a **closure point** of S if, for each $\varepsilon > 0$, $B_\varepsilon(x) \cap S \neq \emptyset$.
- ▶ The **closure** of S , denoted by $\text{cl } S$, is the set of all closure points of S . ▲
- ▶ So x is a closure point of S if every open ball about x contains at least one point in S .

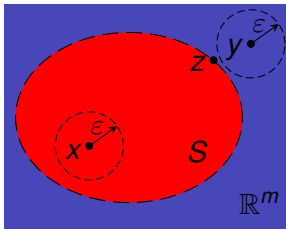


Figure: Points x and z are closure points of the open set S , but point y is not.

Closed Sets.

- ▶ The closure of S is the intersection of all closed sets containing S – this is sometimes used as the definition of the closure of S .
- ▶ By the previous theorem, the closure is a closed set
- ▶ The closure of S is the smallest closed set containing S .
- ▶ Thus a set S is closed in \mathbb{R}^m iff every point in S is a point of closure of S , i.e. iff $S = \text{cl } S$.

Example

Consider the interval (a, b) in \mathbb{R} . The closure of this set is $[a, b]$.

- ▶ The following theorem provides a sequential characterization of the closure of a set

Theorem (S&B 12.11)

Let S be a subset of \mathbb{R}^m . Then $x \in \text{cl } S$ iff there is a sequence (x_n) with $x_n \in S$ for all $n \in \mathbb{N}$ with $\lim x_n = x$.

Closed Sets.

Definition.

Let S be a subset of \mathbb{R}^m .

- ▶ A point $x \in \mathbb{R}^m$ is a **boundary point** of S if, for each $\varepsilon > 0$, $B_\varepsilon(x) \cap S \neq \emptyset$ and $B_\varepsilon(x) \cap S^c \neq \emptyset$.
- ▶ The **boundary** of S is the set of all boundary points of S and is the set $\text{cl } S \setminus \text{int } S$. ▲
- ▶ So x is in the boundary of S if every open ball about x contains both points in S and points in the complement of S .

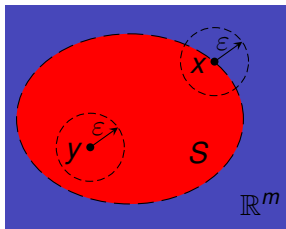


Figure: Point x is a boundary point of the open set S , but point y is not.

Closed Sets.

Example

Consider the intervals (a, b) , $(a, b]$ and $[a, b)$ where $a, b \in \mathbb{R}$. The boundary of each of these sets is $\{a, b\}$. ◆

- ▶ The next theorem is a simple (?) consequence of the definitions of boundary points and closure.

Theorem (S&B 12.12)

Let S be a subset of \mathbb{R}^m . The set of boundary points of S is equal to $\text{cl } S \cap \text{cl } S^c$.

Proof.

A test of your set theory and understanding of closure. ■

Example

Consider the set $S = (a, b]$ where $a, b \in \mathbb{R}$, with $S^c = (-\infty, a] \cup (b, +\infty)$. Then $\text{cl } S = [a, b]$ and $\text{cl } S^c = (-\infty, a] \cup [b, +\infty)$. So $\text{cl } S \cap \text{cl } S^c = \{a, b\}$, the boundary points of S . ◆

Compact Sets.

Definition.

Let S be a subset of \mathbb{R}^m . We say S is **bounded** if there exists a number M such that $\|x\| \leq M$ for all $x \in S$. ▲

- ▶ So a set S in \mathbb{R}^m is bounded if it is contained in some ball in \mathbb{R}^m .

Example

- ▶ Bounded sets include any interval or finite union of intervals in \mathbb{R} except for those with $+\infty$ or $-\infty$ as an endpoint.
- ▶ Any disc in the plane with finite radius is bounded.
- ▶ Sets that are not bounded include the integers in \mathbb{R} , any hyperplane, or any ray. ◆

Compact Sets.

- ▶ We now define compact sets – the generalization of bounded closed intervals to higher dimensions.

Definition.

Let S be a subset of \mathbb{R}^m . We say S is **compact** if it is both closed and bounded ▲

Example

- ▶ Any closed interval is a compact set.
- ▶ Compact sets include any closed disc of finite radius in the plane. Other discs are not. ◆

Compact Sets.

- ▶ The following theorem, called the Bolzano-Weierstrass Theorem states that **any** sequence defined on a compact set must contain a subsequence that converges.
- ▶ It is an important result, as it is used to prove the existence of a maximum and minimum of a continuous function on a compact set – a problem that is important in economics.

Theorem (S&B 12.14)

Let C be a compact subset of \mathbb{R}^m and let (s_n) be a sequence such that $s_n \in C$ for all $n \in \mathbb{N}$. Then (s_n) has a convergent subsequence whose limit is also contained in C .

Proof.

See chapter 29 of Simon and Blume. ■