Metric Space Properties of Euclidean Spaces

Sequences in $\mathbb{R}^m$.

Definition. A *sequence* s in $\mathbb{R}^m$ is any function $s : \mathbb{N} \to \mathbb{R}^m$. ▲

- A sequence in $\mathbb{R}^m$ is an assignment of a vector in $\mathbb{R}^m$ to each natural number.
- For such sequences there are two indices to keep track of:
  - one for the m coordinates of each m-vector, and
  - the other to indicate the element in the sequence.
- We say a sequence in $\mathbb{R}^m$ converges, if as n gets large the terms get "close" to some number s.
- Before we define convergence, we need a way to measure the "closeness" of elements in the sequence.
- We use the notion of distance between vectors we learnt earlier.
- The distance between any two vectors $x, y \in \mathbb{R}^m$ is the norm of their difference:

$$\|x - y\| = \sqrt{(x-y).(x-y)} = \sqrt{(x_1 - y_1)^2 + \cdots + (x_m - y_m)^2}.$$

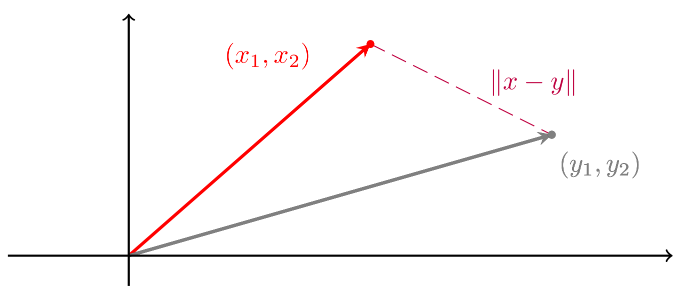

Figure 1: The distance between two vectors in the plane.

Definition. The standard *Euclidean metric* on $\mathbb{R}^m$ is given by

$$d(x,y) = \|x - y\| = \left[\sum_{k=1}^{m} (x_k - y_k)^2\right]^{\frac{1}{2}} \quad ▲$$

- As we saw in our study of Euclidean vector spaces, the Euclidean metric satisfies the following properties.

1. $d(x,x) = 0$ for all $x \in \mathbb{R}^m$ and $d(x,y) > 0$ $x \neq y \in \mathbb{R}^m$.
2. $d(x,y) = d(y,x)$ for all $x, y \in \mathbb{R}^m$.

3. $d(x,z) \le d(x,y) + d(y,z)$ for all $x,y,z \in \mathbb{R}^m$.

- In fact any set S together with a function d satisfying the above properties – called a *distance function* or *metric* – is referred to as a *metric space*.
- The following definition generalizes to $\mathbb{R}^m$ the idea of the ε interval about a point a given by $I_\varepsilon(a) = \{x \in \mathbb{R} \mid |x-a| < \varepsilon\}$.

Definition. Let $\varepsilon > 0$ and let a be a vector in $\mathbb{R}^m$. The *open ε-ball about a* is

$$B_\varepsilon(a) = \{x \in \mathbb{R}^m \mid \|x-a\| < \varepsilon\}.$$ ▲

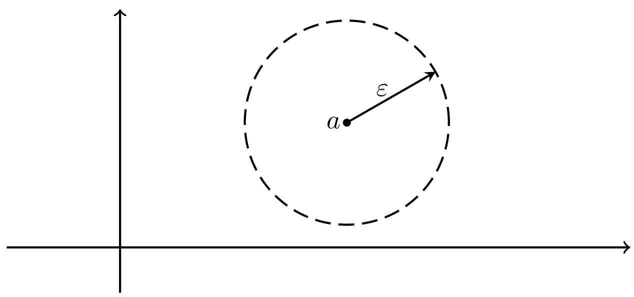

Figure 2: The open ε-ball about $a \in \mathbb{R}^2$.

- So all we have to do is $|\cdot|$ with $\|\cdot\|$ in our definition of convergence.

Definition. A sequence (s_n) in a Euclidean space $\mathbb{R}^m$ *converges* to s if

> for each $\varepsilon > 0$ there exists a number N, such that $n > N$ implies that $s_n \in B_\varepsilon(s)$.

The vector s is called the *limit* of (s_n). ▲

- Thus the sequence of vectors $(s_n)_{n\in\mathbb{N}}$ converges to a limit vector s if the sequence of distances between the vectors s_n and s, given by $(\|s_n - s\|)_{n\in\mathbb{N}}$ converges to 0.
- Equivalently, we say a sequence (s_n) converges to s if $\lim_{n\to+\infty} d(s_n, s) = 0$. (This definition is more general, and applies to any metric space.)

Example 1. Consider the sequence (s_n) in $\mathbb{R}^2$ given by

$$\left(\left(1, \frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{3}\right), \left(-\frac{1}{3}, -\frac{1}{4}\right), \left(-\frac{1}{4}, \frac{1}{5}\right), \left(\frac{1}{5}, \frac{1}{6}\right), \dots\right)$$

The terms in the sequence move clockwise in $\mathbb{R}^2$, as they converge to the origin. Notice that each term s_n gets closer to the origin, each of its components is also getting closer to the origin. ⧫

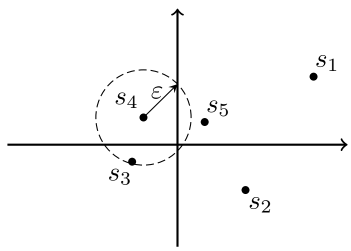

Figure 3: The sequence (s_n) in $\mathbb{R}^2$ converging to 0. For the given ε we need $n > 3$.

- The findings of the last example are true of any convergent sequence in $\mathbb{R}^m$ as the following theorem states.

Theorem 1 (S&B 12.5)**.** *A sequence (s_n) in $\mathbb{R}^m$ converges iff all of its m component sequences converge in $\mathbb{R}$.*

Proof. <2-> We will prove the "only if" part. Let $s_n = (s_{1n}, \dots, s_{mn})$. Suppose that (s_n) converges to $\hat{s}$ and let $\varepsilon > 0$. Then there exists an N such that $n > N$ implies $s_n \in B_\varepsilon(\hat{s})$ or, equivalently, $\|s_n - s\| < \varepsilon$. We want to show any component sequence $(s_{kn})_{n\in\mathbb{N}}$ of $(s_n)_{n\in\mathbb{N}}$ converges to $\hat{s}_k$. Now, for $n > N$ we have

$$\begin{aligned} |s_{kn} - \hat{s}_k| &= \sqrt{(s_{kn} - \hat{s}_k)^2} \leq \sqrt{(s_{1n} - \hat{s}_1)^2 + \cdots + (s_{mn} - \hat{s}_m)^2} \\ &= \|s_n - \hat{s}\| < \varepsilon. \end{aligned}$$

■

- This result reduces the problem of proving convergence of a sequence of vectors in $\mathbb{R}^m$ to the problem of proving the convergence of m sequences in $\mathbb{R}$.
- This allows us to apply results about sequences in $\mathbb{R}$ to sequences in $\mathbb{R}^m$.
  - From the previous theorem and uniqueness of limits of sequences in $\mathbb{R}$ it follows that the limit of a sequence in $\mathbb{R}^m$ is unique.
- The next result generalizes our limit theorem for convergent sequences in $\mathbb{R}$.

Theorem 2 (S&B 12.6)**.** *Suppose that (s_n) and (t_n) are convergent sequences of vectors in $\mathbb{R}^m$, with limits s and t respectively; and let (k_n) be a convergent sequence of real numbers with limit k.*

1. $\lim(k_n s_n) = ks$.
2. $\lim(s_n + t_n) = s + t$.
3. $\lim(s_n \cdot t_n) = s \cdot t$.

Open Sets.

- The next definition formalizes the idea that a set is "open" if it has "no boundary".

Definition. Let S be a subset of $\mathbb{R}^m$.

- We say S is *open* in $\mathbb{R}^m$ if, for each $x \in S$, there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq S$. ▲
- So the set S is open if there exists an open ε-ball about x completely contained in S.
- This means we can always move a small distance in any direction from any point in the set and still be strictly inside the set.

Example 2.

- The set $\mathbb{R}^m$ is open in $\mathbb{R}^m$.
- Any open interval (a, b) in $\mathbb{R}$ is open!
- Any half open interval $(a, b]$ or $[b, a)$ in $\mathbb{R}$ is not an open set.
- Any point in any Euclidean space is not an open set.
- Any line in $\mathbb{R}^2$, any plane in $\mathbb{R}^3$, or any hyperplane in $\mathbb{R}^m$ is not open.
- Any object of lower dimension than the Euclidean space it is in is not open. ⧫

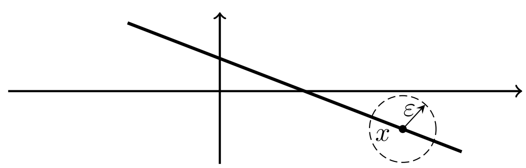

Figure 4: A disc about any point x on a line in $\mathbb{R}^2$ contains points not on the line.

Example 3. Show the interval $(0, 1)$ is an open set.

Proof. Take any point $x \in (0, 1)$. Then $0 < x < 1$ and so

$$0 < \frac{x}{2} = x - \frac{x}{2} < x < 1$$

and

$$0 < x < x + \frac{1-x}{2} = \frac{1+x}{2} < 1$$

So let $\varepsilon = \min\{x/2, (1-x)/2\}$. Then the open ε-interval about x given by $(x - \varepsilon, x + \varepsilon)$ is strictly contained in $(0, 1)$. ■

Theorem 3 (S&B 12.7)**.** *Open balls are open sets.*

Theorem 4 (S&B 12.8)**.**

1. *The union of any collection of open sets is open.*

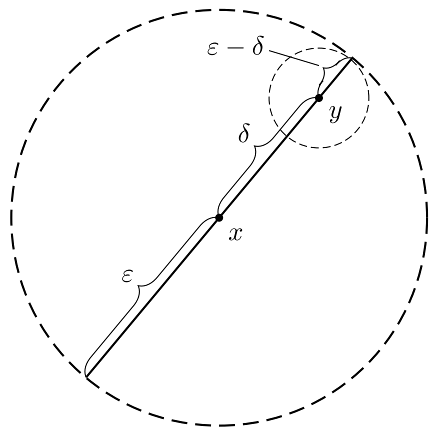

Figure 5: If $y \in B_\varepsilon(x)$ and $\delta = \|x - y\|$, then $B_{\varepsilon-\delta}(y) \subseteq B_\varepsilon(x)$.

2. *The intersection of finitely many open sets is open.*

Proof of (2). <2-> Let $S_1, \ldots, S_n$ be open sets, and let $S = \bigcap_{i=1}^n S_i$. Let $x \in S$.

- To show that S is open, we need to show that there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq S$.
- Now, since $x \in S$ we have $x \in S_i$ for $i = 1, \ldots, n$.
- Since each S_i is open, for each i there exists an ε_i such that $B_{\varepsilon_i}(x) \subseteq S_i$.
- Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. The ball $B_\varepsilon(x)$ is contained in each $B_{\varepsilon_i}(x)$ and so $B_\varepsilon(x) \subseteq S_i$ for all $i = 1, \ldots, n$.

Thus $B_\varepsilon(x) \subseteq S$ and so S is open. ■

Example 4. To see why an intersection of infinitely many open sets need not be open, consider the open intervals $S_n = (-1/n, 1/n)$ in $\mathbb{R}$ for $n \in \mathbb{N}$.

- A set S_n is simply an interval of radius $1/n$ about 0.
- The intersection of these sets is $\bigcap_{n\in\mathbb{N}} S_n = \{0\}$ since if $y \neq 0$, $y \notin S_n$ for all $n > 1/y$.
- But a point can never be open, so the infinite intersection of the sets S_n is not open. ♦

Definition. Let S be a subset of $\mathbb{R}^m$.

- A point $x \in \mathbb{R}^m$ is an *interior point* of S if, for some $\varepsilon > 0$, $B_\varepsilon(x) \subseteq S$.
- The *interior* of S, denoted by $\operatorname{int} S$ is the set of all interior points of S. ▲
- So x is an interior point of S if there is some open ball about x contained in S.

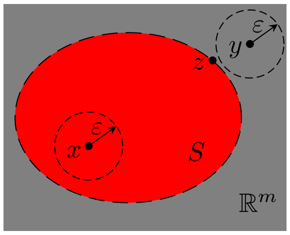

Figure 6: Point x is an interior point of the open set S, but points y and z are not.

- The interior of S is the union of all open sets contained in S – this is sometimes used as the definition of the interior of S.
- By the previous theorem, the interior is an open set.
- The interior of S is the largest open set contained in S.
- Thus a set S is open in $\mathbb{R}^m$ iff every point in S is interior to S, i.e. iff $S = \operatorname{int} S$.

Example 5.

- Consider the interval $[a, b]$ in $\mathbb{R}$.
- A point x is interior to $[a, b]$ if for some $\varepsilon > 0$, the set $\{y \in \mathbb{R} \mid |x - y| < \varepsilon\}$ is a subset of $[a, b]$.
- That is, x is interior if for all y such that $x - \varepsilon < y < x + \varepsilon$, we have $a \leq y \leq b$.
- Thus we need $a \leq x - \varepsilon$ and $x + \varepsilon \leq b$ for some $\varepsilon > 0$.
- Hence a point x is interior if $a < x < b$ i.e. if $x \in (a, b)$. ⧫

Closed Sets.

- The next definition formalizes the idea that set is closed if it contains all its "boundary points."

Definition. Let S be a subset of $\mathbb{R}^m$. We say S is *closed* in $\mathbb{R}^m$ if, whenever (s_n) is a convergent sequence such that $s_n \in S$ for all $n \in N$, its limit is also contained in S. ▲

- Thus a set is closed if it contains all its limit points.
- This means that for any given point x, if there are points in the set arbitrarily close to x, then x must be in the set too.
- Closedness and openness are complementary properties – a closed set contains its boundary points, an open set does not.
- We will use this property to provide another definition of a closed set. But first we define the complement of a set.

Definition. Let S be a subset of $\mathbb{R}^m$. The *complement* of S, denoted S^c set defined by

$$S^c = \{x \in \mathbb{R}^m \mid x \notin S\} = \mathbb{R}^m \setminus S. \qquad \blacktriangle$$

- Thus the complement of S is the set of all points that are not in S.
- We now state the result that the complement of an open set is closed and the complement of a closed set is open.

Theorem 5 (S&B 12.9)**.** *A subset S of $\mathbb{R}^m$ is closed iff if its complement S^c is open.*

- This theorem provides an alternative definition of a closed set as a set whose complement is open.

Example 6.

- Any set of the form $(-\infty, a] \cup [b, +\infty)$ is closed, since this set is the complement of the set (a, b).
- The set $\mathbb{R}^m$ is closed in $\mathbb{R}^m$ because any convergent sequence in $\mathbb{R}^m$ has a limit in $\mathbb{R}^m$. Note that $\mathbb{R}^m$ is both closed and open.
- The empty set is also closed and open in $\mathbb{R}^m$. It is the complement of $\mathbb{R}^m$. ⧫

Theorem 6 (S&B 12.10)**.**

1. *The intersection of any collection of closed sets is closed.*
2. *The union of finitely many closed sets is closed.*

Proof. Use theorem (5), theorem (4) and the fact that $(\bigcup_i S_i)^c = \bigcap_i (S_i^c)$. ■

Example 7. To see why a union of infinitely many closed sets need not be closed, consider the closed intervals $S_n = [-n/(n+1), n/(n+1)]$ in $\mathbb{R}$ for $n \in \mathbb{N}$.

- The set S_n is the interval of radius $n/(n+1)$ about 0.
- The union of these sets is $\bigcup_{n \in \mathbb{N}} S_n = (-1, 1)$.
- To show this, consider any $y \in \bigcup_{n \in \mathbb{N}} S_n$. Then $y \in S_n$ for some $n \in \mathbb{N}$ and so $-n/(n+1) \leq y \leq n(n+1)$ for some n.
- It follows that $y \in (-1, 1)$ and that $\bigcup_{n \in \mathbb{N}} S_n \subseteq (-1, 1)$.
- Now, let $y \in (-1, 1)$. Then $-1 < y < 1$, and so we can find an $n \in N$ such that $-n/(n+1) \leq y \leq n(n+1)$.
- Thus $(-1, 1) \subseteq \bigcup_{n \in \mathbb{N}} S_n$.
- But the set $(-1, 1)$ is open. ⧫

Definition. Let S be a subset of $\mathbb{R}^m$.

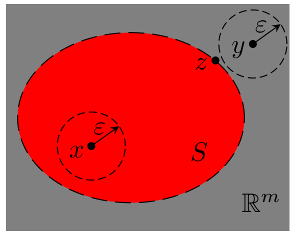

Figure 7: Points x and z are closure points of the open set S, but point y is not.

- A point $x \in \mathbb{R}^m$ is a *closure point* of S if, for each $\varepsilon > 0$, $B_\varepsilon(x) \cap S \neq 0$.
- The *closure* of S, denoted by $\mathrm{cl}\, S$, is the set of all closure points of S. ▲
- So x is a closure point of S if every open ball about x contains at least one point in S.

- The closure of S is the intersection of all closed sets containing S – this is sometimes used as the definition of the closure of S.
- By the previous theorem, the closure is a closed set
- The closure of S is the smallest closed set containing S.
- Thus a set S is closed in $\mathbb{R}^m$ iff every point in S is a point of closure of S, i.e. iff $S = \mathrm{cl}\, S$.

Example 8. Consider the interval (a, b) in $\mathbb{R}$. The closure of this set is $[a, b]$.

- The following theorem provides a sequential characterization of the closure of a set

Theorem 7 (S&B 12.11)**.** *Let S be a subset of $\mathbb{R}^m$. Then $x \in \mathrm{cl}\, S$ iff there is a sequence (x_n) with $x_n \in S$ for all $n \in \mathbb{N}$ with $\lim x_n = x$.*

Definition. Let S be a subset of $\mathbb{R}^m$.

- A point $x \in \mathbb{R}^m$ is a *boundary point* of S if, for each $\varepsilon > 0$, $B_\varepsilon(x) \cap S \neq \emptyset$ and $B_\varepsilon(x) \cap S^c \neq \emptyset$.
- The *boundary* of S is the set of all boundary points of S and is the set $\mathrm{cl}\, S \setminus \mathrm{int}\, S$. ▲
- So x is in the boundary of S if every open ball about x contains both points in S and points in the complement of S.

Example 9. Consider the intervals (a, b), $(a, b]$ and $[a, b)$ where $a, b \in \mathbb{R}$. The boundary of each of these sets is $\{a, b\}$. ⧫

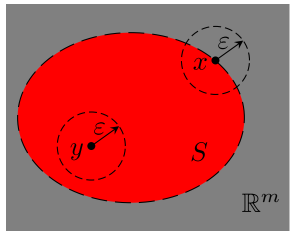

Figure 8: Point x is a boundary point of the open set S, but point y is not.

- The next theorem is a simple (?) consequence of the definitions of boundary points and closure.

Theorem 8 (S&B 12.12)**.** *Let S be a subset of $\mathbb{R}^m$. The set of boundary points of S is equal to* $\operatorname{cl} S \cap \operatorname{cl} S^c$.

Proof. A test of your set theory and understanding of closure. ■

Example 10. Consider the set $S = (a, b]$ where $a, b \in \mathbb{R}$, with $S^c = (-\infty, a] \cup (b, +\infty)$. Then $\operatorname{cl} S = [a, b]$ and $\operatorname{cl} S^c = (-\infty, a] \cup [b, +\infty)$. So $\operatorname{cl} S \cap \operatorname{cl} S^c = \{a, b\}$, the boundary points of S. ⧫

Compact Sets.

Definition. Let S be a subset of $\mathbb{R}^m$. We say S is *bounded* if there exists a number M such that $\|x\| \leq M$ for all $x \in S$. ▲

- So a set S in $\mathbb{R}^m$ is bounded if it is contained in some ball in $\mathbb{R}^m$.

Example 11.

- Bounded sets include any interval or finite union of intervals in $\mathbb{R}$ except for those with $+\infty$ or $-\infty$ as an endpoint.
- Any disc in the plane with finite radius is bounded.
- Sets that are not bounded include the integers in $\mathbb{R}$, any hyperplane, or any ray. ⧫

- We now define compact sets – the generalization of bounded closed intervals to higher dimensions.

Definition. Let S be a subset of $\mathbb{R}^m$. We say S is *compact* if it is both closed and bounded ▲

Example 12.

- Any closed interval is a compact set.

- Compact sets include any closed disc of finite radius in the plane. Other discs are not. ♦

- The following theorem, called the Bolzano-Weierstrass Theorem states that *any* sequence defined on a compact set must contain a subsequence that coverges.

- It is an important result, as it is used to prove the existence of a maximum and minimum of a continuous function on a compact set – a problem that is important in economics.

Theorem 9 (S&B 12.14)**.** *Let C be a compact subset of $\mathbb{R}^m$ and let (s_n) be a sequence such that $s_n \in C$ for all $n \in \mathbb{N}$. Then (s_n) has a convergent subsequence whose limit is also contained in C.*

Proof. See chapter 29 of Simon and Blume. ■