Systems of Linear Equations

Systems of Linear Equations.

- We consider the problem of solving linear systems of equations, such as

$$\begin{aligned} x_1 - 2x_2 &= 8 \\ 3x_1 + x_2 &= 3 \end{aligned}$$

- In general, we write a system of m equations in n unknowns as

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{gathered}$$

- We can solve such systems using one of three types of methods
  1. substitution (read S&B 7.1)
  2. elimination of variables (read S&B 7.1)
  3. matrix methods

Elementary Row Operations.

- We can simplify the representation of a linear system using *matrices*. The coefficients a_{ij} can be arranged in an array

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

  which is called the *coefficient matrix*.

- We can represent the entire linear system using the *augmented matrix*:

$$\hat{A} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right).$$

- There are three *elementary row operations* we can perform on a matrix:
  1. interchange two rows of a matrix,

2. add a multiple of a row to another row, and
3. multiply each element in a row by the same number.

Performing any of these operations on an augmented matrix gives a new matix which represents the same system of equations as the old matrix.

- These operations are equivalent to those used when applying the method of elimination to solve a system. As we will see, we can use these operations to solve a system of linear equations. But first some definitions.

Definition. A row of a matrix is said to have k *leading zeros* if the first k elements of the row are all zeros and the $(k+1)$th element of the row is not zero. A matrix is in *row echelon form* if each row has more leading zeros than the row preceding it. ▲

Example 1. The matrices

$$\begin{pmatrix} 5 & 0 & 6 \\ 0 & -2 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 4 & -3 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 4 & -8 & 4 \\ 0 & -2 & 6 \\ 0 & 0 & 2 \end{pmatrix},$$

are all in row echelon form. ♦

Definition. The first nonzero entry in each row of a matrix in row echelon form is called a *pivot*. A row echelon matrix is in *reduced row echelon form* if each pivot is a 1, and each column containing a pivot has no other nonzero entries. ▲

Example 2. Continuing our last example,

$$\begin{pmatrix} 1 & 0 & 6/5 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1/2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

are those matrices in reduced row echelon form. ♦

- We now look at *Gauss-Jordan Elimination*. This is the process of reducing a matrix to reduced row echelon form.

Example 3.

- Consider the linear system of 3 equations in 3 unknowns:

$$\begin{aligned} -3x_2 + 9x_3 &= 0 \\ 2x_1 - 2x_2 - 2x_3 &= -6 \\ -4x_1 - 2x_2 + 17x_3 &= 22 \end{aligned}$$

- We can represent this compactly as an augmented matrix

$$\left(\begin{array}{rrr|r} 0 & -3 & 9 & 0 \\ 2 & -2 & -2 & -6 \\ -4 & -2 & 17 & 22 \end{array}\right).$$

- We first reduce this to row echelon form:

$$\left(\begin{array}{rrr|r} 0 & -3 & 9 & 0 \\ 2 & -2 & -2 & -6 \\ -4 & -2 & 17 & 22 \end{array}\right) \sim \begin{array}{c} \rho_2 \\ \rho_1 \\ \end{array} \left(\begin{array}{rrr|r} 2 & -2 & -2 & -6 \\ 0 & -3 & 9 & 0 \\ -4 & -2 & 17 & 22 \end{array}\right)$$

$$\sim_{\rho_3+2\rho_1} \left(\begin{array}{rrr|r} 2 & -2 & -2 & -6 \\ 0 & -3 & 9 & 0 \\ 0 & -6 & 13 & 10 \end{array}\right) \sim_{\rho_3-2\rho_2} \left(\begin{array}{rrr|r} 2 & -2 & -2 & -6 \\ 0 & -3 & 9 & 0 \\ 0 & 0 & -5 & 10 \end{array}\right)$$

- Now we obtain the reduced row echelon form:

$$\sim \begin{array}{r} \rho_1/2 \\ -\rho_2/3 \\ -\rho_3/5 \end{array} \left(\begin{array}{rrr|r} 1 & -1 & -1 & -3 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right) \sim \begin{array}{r} \rho_1+\rho_2 \\ \rho_2+3\rho_3 \\ \end{array} \left(\begin{array}{rrr|r} 1 & 0 & -4 & -3 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

$$\sim \begin{array}{r} \rho_1+\rho_2 \\ \rho_2+3\rho_3 \\ \end{array} \left(\begin{array}{rrr|r} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & -2 \end{array}\right)$$

- So the solution to the linear system is $x = (-11, -6, -2)$, i.e. $x_1 = -11$, $x_2 = -6$, $x_3 = -2$. ♦

Systems with Many or No Solutions.

- Systems can have one, many or no solutions (see S&B 7.3).
- The locus of points satisfying a linear equation is a straight line in a plane. So a solution of a system of two linear equations in two unknowns

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2,$$

  is a point which lies on both lines.

Example 4.

- If the lines coincide, as in the system

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 2x_1 + 4x_2 &= 6, \end{aligned}$$

then there are infinitely many solutions.

- If the lines are parallel and do not coincide, as in the system

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_1 + 2x_2 &= 4, \end{aligned}$$

then there is no solution.

- Finally, if the lines cross, as in the system

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_1 + x_2 &= 4, \end{aligned}$$

the solution is unique ($x_1 = 5$, $x_2 = -1$). ♦

Example 5. Consider the linear system with augmented matrix:

$$\left(\begin{array}{cccccc} 1 & 4 & 17 & 4 & | & 38 \\ 2 & 12 & 46 & 10 & | & 98 \\ 3 & 18 & 69 & 17 & | & 153 \end{array} \right).$$

- The reduced row echelon form is

$$\left(\begin{array}{cccccc} 1 & 0 & 5 & 0 & | & 10 \\ 0 & 1 & 3 & 0 & | & 4 \\ 0 & 0 & 0 & 1 & | & 3 \end{array} \right),$$

from which we can read off the solution

$$\begin{array}{rcl} x_1 \qquad + 5x_3 \qquad &=& 10 \\ x_2 + 3x_3 \qquad &=& 4 \\ x_4 &=& 3. \end{array}$$

- This can be rewritten as

$$\begin{aligned} x_1 &= 10 - 5x_3 \\ x_2 &= 4 - 3x_3 \\ x_4 &= 3. \end{aligned}$$

- Notice that x_4 is determined unambiguously, but the other variables are not. The variable x_3 is a *free variable* and can take any value. The other two varaibles x_1 and x_2 are then determined by the above equations. x_1, x_2 and x_4 are *basic variables*. ♦

- In general a variable x_j is basic (free) if the jth column of the row echelon matrix contains (does not contain) a pivot.
- Which variables are free and which are basic may depend on the order of the operations used in the Gaussian elimination process, and the order in which the variables are indexed.

Rank – The Fundamental Criterion.

- The main criterion for answering questions about existence and uniqueness of solutions is the rank of a matrix.

Definition. The *rank* of a matrix is the number of nonzero rows in its row echelon form. ▲

- The ranks of a coefficient matrix A and augmented matrix $\hat{A}$ satisfy (Fact 7.1 S&B):
  - $\operatorname{rank} A \leq \operatorname{rank} \hat{A}$,
  - $\operatorname{rank} A \leq$number of rows of A, and
  - $\operatorname{rank} A \leq$number of columns of A.

Definition. A matrix A is said to have *maximal rank* if

$$\operatorname{rank} A = \min\{\text{number of rows, number of columns}\}. \qquad ▲$$

Example 6. Consider the following matrices

$$\begin{pmatrix} 5 & 0 & 6 \\ 0 & -2 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 4 & -3 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 4 & -8 & 4 \\ 0 & -2 & 6 \\ 0 & 0 & 2 \end{pmatrix}.$$

These matrices have ranks 2, 2 and 3 respectively. The last two matrices have maximal rank. ♦

Theorem 1. *Let A be the coefficient matrix and $\hat{A}$ the augmented matrix of some linear system $Ax = b$.*

1. *A system has a solution iff* $\operatorname{rank} \hat{A} = \operatorname{rank} A$ *(Fact 7.2 S&B).*
2. *A system must have either no solution, one solution, or infinitely many solutions. Thus, if a system has more than one solution, then it has infinitely many (Fact 7.3 S&B).*
3. *If a system has exactly one solution, then A has at least as many rows (equations) as columns (unknowns) (Fact 7.4 S&B).*
4. *If a system has more unknowns than equations, it must have no solution or infinitely many solutions (Fact 7.5 S&B).*

5. *If a homogeneous system ($b = 0$) has more unknowns than equations, it must have infinitely many solutions. (Fact 7.6 S&B).*

Theorem 2. *Let A be the coefficient matrix of some linear system $Ax = b$.*

1. *A system will have a solution for every choice of b iff*

$$\text{rank}\, A = \textit{number of rows of } A \quad \textit{(Fact 7.7 S\&B).}$$

2. *A system will have at most one solution for every choice of b iff*

$$\text{rank}\, A = \textit{number of columns of } A \quad \textit{(Fact 7.9 S\&B).}$$

3. *A matrix A is nonsingular i.e. a system has one and only one solution for every choice of b iff*

$$\begin{aligned} \text{rank}\, A &= \textit{number of rows of } A \\ &= \textit{number of columns of } A \quad \textit{(Fact 7.10 S\&B).} \end{aligned}$$

Theorem 3 (Fact 7.11 S&B)**.** *Consider the linear system of equations $Ax = b$.*

1. *If the number of equations $<$ the number of unknowns, then:*
   (a) *$Ax = 0$ has infinitely many solutions,*
   (b) *for any given b, $Ax = b$ has 0, or infinitely many solutions, and*
   (c) *if $\text{rank}\, A =$ number of equations, $Ax = b$ has infinitely many solutions for* every *b.*
2. *If the number of equations $>$ the number of unknowns, then:*
   (a) *$Ax = 0$ has one or infinitely many solutions,*
   (b) *for any given b, $Ax = b$ has 0, 1, or infinitely many solutions, and*
   (c) *if $\text{rank}\, A =$ number of unknowns, $Ax = b$ has 0 or 1 solution for* every *b.*
3. *If the number of equations $=$ the number of unknowns, then:*
   (a) *$Ax = 0$ has one or infinitely many solutions,*
   (b) *for any given b, $Ax = b$ has 0, 1, or infinitely many solutions, and*
   (c) *if $\text{rank}\, A =$ number of unknowns, $Ax = b$ has exactly 1 solution for* every *b.*