MTAEA – Linear Independence

Scott McCracken

School of Economics,
Australian National University

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Linear Independence.

Definition.

Let $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$ be a set of vectors.

- The vectors are linearly dependent if there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$, not all zero, such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k = 0.$$

- The vectors are linearly independent if $\lambda_1 v_1 + \cdots + \lambda_k v_k = 0$ for scalars $\lambda_1, \ldots, \lambda_k$ implies that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0.$$

▲

Linear Independence.

Example

(1)
- The vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

  in $\mathbb{R}^n$ are linearly independent.
- To show this, consider any scalars $\lambda_1, \ldots, \lambda_n$ such that $\lambda_1 e_1 + \lambda_2 e_2 \cdots + \lambda_n e_n = 0$, i.e. such that

$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \lambda_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

- The last equation implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

Linear Independence.

(2)

- The vectors

$$w_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, w_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \text{ and } w_n = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

  in $\mathbb{R}^3$ are linearly dependent, since $w_1 - 2w_2 + w_3 = 0$.
- How did we come up with this? Start with the equation in the definition of linear dependence

$$\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_3 \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

  and solve this system for possible values of $\lambda_1, \lambda_2, \lambda_3$.

Linear Independence.

- We can write this system in matrix form as

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

- The coefficient matrix A is simply the matrix with columns w_1, w_2, w_3 which we can write as $A = (w_1 w_2 w_3)$. If we reduce A to row echelon form we get

$$\begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}.$$

- We have found that rank A is not maximal, and so the system is singular and has a nonzero solution (many in fact). ♦

- The following theorem generalizes the last example.

Linear Independence.

Theorem (S&B 11.1)

Vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ are linearly dependent iff the linear system

$$A \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} = 0,$$

where $A = (v_1 v_2 \cdots v_k)$, has a nonzero solution $(\lambda_1, \ldots, \lambda_k)$.

- The following theorem gives us a way of checking that a set of n vectors in $\mathbb{R}^n$ are linearly independent.

Theorem (S&B 11.2)

Vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ are linearly independent iff

$$\det(v_1 v_2 \cdots v_n) \neq 0.$$

Linear Independence.

Theorem (S&B 11.3)

If $k > n$, any set of k vectors in $\mathbb{R}^n$ is linearly dependent.

Proof.

Let $v_1, \ldots, v_k$ be k vectors in $\mathbb{R}^n$ with $k > n$. By theorem 1, this set of vectors is linearly dependent if and only if the system

$$A\lambda = A \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} = 0$$

where $A = (v_1 \, v_2 \cdots v_k)$, has a nonzero solution $(\lambda_1, \ldots, \lambda_k)$.

- But, since the vectors belong to $\mathbb{R}^n$, the matrix A has n rows and k columns with $k > n$.

Linear Independence.

- Since A has more columns than rows, there must be some column, say column j, in the row echelon forms of A without a pivot.
- But then λ_j is a free variable, and since $A\lambda = 0$ has a solution, it must have infinitely many solutions all but one of which is nonzero.
- Hence the vectors are linearly dependent. ■
- Instead of the above, we could have used fact 7.11(a)(i) of S&B which says that if there are more unknowns than equations, the system $A\lambda = 0$ must have infinitely many solutions.

Spanning Sets.

- A subset of V of the vector space $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if it is
  - closed under addition (the sum of any two elements in V is also in V), and
  - closed under scalar multiplication (any scalar multiple of an element in V is also in V).

Example

- The set $\{(1,1,1)\}$ is not a subspace of $\mathbb{R}^3$, because for example $(0,0,0) \notin \{(1,1,1)\}$.
- The set $V = \mathcal{L}[(1,1,1)] = \{\lambda(1,1,1) \mid \lambda \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$ as you can check.

Spanning Sets.

Definition.

Let $v_1, \ldots, v_k \in \mathbb{R}^n$ be a set of vectors and let $V \subseteq \mathbb{R}^n$ be a subspace.

- The set of all linear combinations of these vectors

$$\mathcal{L}[v_1, \ldots, v_k] = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \mid \lambda_1, \ldots, \lambda_k \in \mathbb{R}\}$$

  is called the set generated or spanned by $v_1, \ldots, v_k$.
- Equivalently, the span of $\{v_1, \ldots, v_k\}$ is the set $\mathcal{L}[v_1, \ldots, v_k]$ (sometimes written span $(v_1, \ldots, v_k)$).
- If

$$V = \mathcal{L}[v_1, \ldots, v_k],$$

  we say $v_1, \ldots, v_k$ span V. ▲

- A set of vectors spans (or is a spanning set of) a subspace V of $\mathbb{R}^n$ if we can write any element in V as a linear combination of the vectors.

Spanning Sets.

Example

(1) The n-dimensional Euclidean space is spanned by the vectors $e_1, \ldots, e_n$ of our earlier example. Take any vector $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Then $a = a_1 e_1 + \cdots + a_n e_n$.

(2) There are many sets of vectors that span the same space. Each of the following sets spans the space $\mathbb{R}^2$.

(a) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(b) $\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(c) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 56 \\ -24 \end{pmatrix}$.

(d) $\begin{pmatrix} 6 \\ -6 \end{pmatrix}, \begin{pmatrix} 31 \\ 31 \end{pmatrix}$.

Spanning Sets.

Theorem (S&B 11.4)

Let $v_1, \dots v_k \in \mathbb{R}^n$ be a set of vectors. Let $A = (v_1 v_2 \cdots v_k)$ and let $b \in \mathbb{R}^n$ be a vector. Then b lies in the space $\mathcal{L}[v_1 \dots, v_k]$ spanned by $v_1, \dots, v_k$ iff the system $A\lambda = b$ has a solution λ.

- The first corollary to this theorem provides a way of checking whether or not a set of vectors spans all of $\mathbb{R}^n$, while the second gives the minimum number of vectors needed in a set spanning $\mathbb{R}^n$.

Corollary (S&B 11.5)

Let $v_1, \dots v_k \in \mathbb{R}^n$ be a set of vectors. Let $A = (v_1 v_2 \cdots v_k)$. Then $v_1, \dots, v_k$ span $\mathbb{R}^n$ iff the system of equations $Ax = b$ has a solution x for every $b \in \mathbb{R}^n$.

Corollary (S&B 11.6)

A set of vectors that spans $\mathbb{R}^n$ must contain at least n vectors.

Basis and Dimension.

- Once we have a spanning set of vectors, we can always make the spanning set larger by including more vectors.
- We want to find the smallest set of vectors that spans a subset V of the n-dimensional Euclidean space.

Definition.

Let $v_1, \ldots v_k \in V$ be a set of vectors and let $V \subseteq \mathbb{R}^n$. Then $v_1, \ldots v_k$ forms a basis of V if

(1) $v_1, \ldots v_k$ span V, and

(2) $v_1, \ldots v_k$ are linearly independent. ▲

Basis and Dimension.

Example

(1) From our previous examples, we can see that the unit vectors $e_1, \ldots, e_n$ form a basis of $\mathbb{R}^n$. This natural basis, is called the canonical basis of $\mathbb{R}^n$.

(2) In a previous example (2), we listed four spanning sets of $\mathbb{R}^2$. The set (c) is not linearly independent, since it contains more than two vectors. The collections (a), (b), and (d) are linearly independent and so each forms a basis for $\mathbb{R}^2$. ♦

Theorem (S&B 11.7)

Every basis of $\mathbb{R}^n$ contains exactly n vectors.

Basis and Dimension.

- Using theorems 1, 2 and corollary 4, together with the fact that a square matrix is nonsingular iff its determinant is nonzero gives the following theorem.

Theorem (S&B 11.8)

Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be a set of vectors and let $A = (v_1 v_2 \cdots v_n)$. Then the following statements are equivalent.

(1) $v_1, \ldots, v_n$ *are linearly independent.*

(2) $v_1, \ldots, v_n$ *span* $\mathbb{R}^n$*.*

(3) $v_1, \ldots, v_n$ *form a basis of* $\mathbb{R}^n$*.*

(4) $|A| \neq 0$.

(5) $\operatorname{rank} A = n$.

Basis and Dimension.

- The fact that any basis of $\mathbb{R}^n$ contains exactly n vectors tells us that there are n independent directions in $\mathbb{R}^n$.
- This is why we say that $\mathbb{R}^n$ is n-dimensional.

Definition.

The number of vectors in any basis of a subspace V is called the dimension of V. ▲

- This definition only makes sense if any basis of V has the same number of vectors. Fortunately, this can be shown to be true (see theorem 27.3 of S&B if you are interested).