Linear Independence

Linear Independence.

Definition. Let $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$ be a set of vectors.

- The vectors are *linearly dependent* if there exist scalars $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{R}$, not all zero, such that
$$\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k = 0.$$
- The vectors are *linearly independent* if $\lambda_1 v_1 + \cdots + \lambda_k v_k = 0$ for scalars $\lambda_1, \ldots, \lambda_k$ implies that
$$\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0.$$ ▲

Example 1.

1. - The vectors
$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$
in $\mathbb{R}^n$ are linearly independent.
   - To show this, consider any scalars $\lambda_1, \ldots, \lambda_n$ such that $\lambda_1 e_1 + \lambda_2 e_2 \cdots + \lambda_n e_n = 0$, i.e. such that
$$\lambda_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \lambda_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$
   - The last equation implies that $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.
2. - The vectors
$$w_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, w_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \text{ and } w_n = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$
in $\mathbb{R}^3$ are linearly dependent, since $w_1 - 2w_2 + w_3 = 0$.
   - How did we come up with this? Start with the equation in the definition of linear dependence
$$\lambda_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_3 \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
and solve this system for possible values of $\lambda_1, \lambda_2, \lambda_3$.

2.
   - We can write this system in matrix form as

$$\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

   - The coefficient matrix A is simply the matrix with columns w_1, w_2, w_3 which we can write as $A = (w_1 w_2 w_3)$. If we reduce A to row echelon form we get

$$\begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}.$$

   - We have found that $\operatorname{rank} A$ is not maximal, and so the system is singular and has a nonzero solution (many in fact). ⧫

- The following theorem generalizes the last example.

Theorem 1 (S&B 11.1)**.** *Vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ are linearly dependent iff the linear system*

$$A \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} = 0,$$

where $A = (v_1 v_2 \cdots v_k)$, has a nonzero solution $(\lambda_1, \ldots, \lambda_k)$.

- The following theorem gives us a way of checking that a set of n vectors in $\mathbb{R}^n$ are linearly independent.

Theorem 2 (S&B 11.2)**.** *Vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ are linearly independent iff*

$$\det(v_1 v_2 \cdots v_n) \neq 0.$$

Theorem 3 (S&B 11.3)**.** *If $k > n$, any set of k vectors in $\mathbb{R}^n$ is linearly dependent.*

Proof. Let $v_1, \ldots, v_k$ be k vectors in $\mathbb{R}^n$ with $k > n$. By theorem 1, this set of vectors is linearly dependent if and only if the system

$$A\lambda = A \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} = 0$$

where $A = (v_1 v_2 \cdots v_k)$, has a nonzero solution $(\lambda_1, \ldots, \lambda_k)$.

- But, since the vectors belong to $\mathbb{R}^n$, the matrix A has n rows and k columns with $k > n$.

- Since A has more columns than rows, there must be some column, say column j, in the row echelon forms of A without a pivot.
- But then λ_j is a free variable, and since $A\lambda = 0$ has a solution, it must have infinitely many solutions all but one of which is nonzero.
- Hence the vectors are linearly dependent. ■
- Instead of the above, we could have used fact 7.11(a)(i) of S&B which says that if there are more unknowns than equations, the system $A\lambda = 0$ must have infinitely many solutions.

Spanning Sets.

- A subset of V of the vector space $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if it is
  - closed under addition (the sum of any two elements in V is also in V), and
  - closed under scalar multiplication (any scalar multiple of an element in V is also in V).

  Example 2.

  - The set $\{(1, 1, 1)\}$ is not a subspace of $\mathbb{R}^3$, because for example $(0, 0, 0) \notin \{(1, 1, 1)\}$.
  - The set $V = \mathcal{L}[(1, 1, 1)] = \{\lambda(1, 1, 1) \mid \lambda \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^3$ as you can check.

Definition. Let $v_1, \ldots, v_k \in \mathbb{R}^n$ be a set of vectors and let $V \subseteq \mathbb{R}^n$ be a subspace.

- The set of all *linear combinations* of these vectors

$$\mathcal{L}[v_1, \ldots, v_k] = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \mid \lambda_1, \ldots, \lambda_k \in \mathbb{R}\}$$

  is called the set *generated* or *spanned* by $v_1, \ldots, v_k$.
- Equivalently, the *span* of $\{v_1, \ldots, v_k\}$ is the set $\mathcal{L}[v_1, \ldots, v_k]$ (sometimes written $\mathrm{span}\,(v_1, \ldots, v_k)$).
- If

$$V = \mathcal{L}[v_1, \ldots, v_k],$$

  we say $v_1, \ldots, v_k$ *span* V. ▲
- A set of vectors spans (or is a *spanning set* of) a subspace V of $\mathbb{R}^n$ if we can write any element in V as a linear combination of the vectors.

Example 3.

1. The n-dimensional Euclidean space is spanned by the vectors $e_1, \ldots, e_n$ of our earlier example. Take any vector $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Then $a = a_1 e_1 + \cdots + a_n e_n$.
2. There are many sets of vectors that span the same space. Each of the following sets spans the space $\mathbb{R}^2$.

(a) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(b) $\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(c) $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 56 \\ -24 \end{pmatrix}$.

(d) $\begin{pmatrix} 6 \\ -6 \end{pmatrix}, \begin{pmatrix} 31 \\ 31 \end{pmatrix}$.

Theorem 4 (S&B 11.4)**.** *Let $v_1, \dots v_k \in \mathbb{R}^n$ be a set of vectors. Let $A = (v_1 v_2 \cdots v_k)$ and let $b \in \mathbb{R}^n$ be a vector. Then b lies in the space $\mathcal{L}[v_1 \dots, v_k]$ spanned by $v_1, \dots, v_k$ iff the system $A\lambda = b$ has a solution λ.*

- The first corollary to this theorem provides a way of checking whether or not a set of vectors spans all of $\mathbb{R}^n$, while the second gives the minimum number of vectors needed in a set spanning $\mathbb{R}^n$.

Corollary 1 (S&B 11.5)**.** *Let $v_1, \dots v_k \in \mathbb{R}^n$ be a set of vectors. Let $A = (v_1 v_2 \cdots v_k)$. Then $v_1, \dots, v_k$ span $\mathbb{R}^n$ iff the system of equations $Ax = b$ has a solution x for every $b \in \mathbb{R}^n$.*

Corollary 2 (S&B 11.6)**.** *A set of vectors that spans $\mathbb{R}^n$ must contain at least n vectors.*

Basis and Dimension.

- Once we have a spanning set of vectors, we can always make the spanning set larger by including more vectors.
- We want to find the smallest set of vectors that spans a subset V of the n-dimensional Euclidean space.

Definition. Let $v_1, \dots v_k \in V$ be a set of vectors and let $V \subseteq \mathbb{R}^n$. Then $v_1, \dots v_k$ forms a *basis* of V if

1. $v_1, \dots v_k$ span V, and
2. $v_1, \dots v_k$ are linearly independent. ▲

Example 4.

1. From our previous examples, we can see that the unit vectors $e_1, \dots, e_n$ form a basis of $\mathbb{R}^n$. This natural basis, is called the *canonical basis* of $\mathbb{R}^n$.
2. In a previous example (2), we listed four spanning sets of $\mathbb{R}^2$. The set (2c) is not linearly independent, since it contains more than two vectors. The collections (2a), (2b), and (2d) are linearly independent and so each forms a basis for $\mathbb{R}^2$. ⧫

Theorem 5 (S&B 11.7)**.** *Every basis of $\mathbb{R}^n$ contains exactly n vectors.*

- Using theorems 1, 2 and corollary 1, together with the fact that a square matrix is nonsingular iff its determinant is nonzero gives the following theorem.

Theorem 6 (S&B 11.8)**.** *Let $v_1, \dots, v_n \in \mathbb{R}^n$ be a set of vectors and let $A = (v_1 v_2 \cdots v_n)$. Then the following statements are equivalent.*

1. $v_1, \ldots, v_n$ *are linearly independent.*

2. $v_1, \ldots, v_n$ *span* $\mathbb{R}^n$.

3. $v_1, \ldots, v_n$ *form a basis of* $\mathbb{R}^n$.

4. $|A| \neq 0$.

5. $\operatorname{rank} A = n$.

- The fact that any basis of $\mathbb{R}^n$ contains exactly n vectors tells us that there are n independent directions in $\mathbb{R}^n$.

- This is why we say that $\mathbb{R}^n$ is n-dimensional.

Definition. The number of vectors in any basis of a subspace V is called the *dimension* of V. ▲

- This definition only makes sense if any basis of V has the same number of vectors. Fortunately, this can be shown to be true (see theorem 27.3 of S&B if you are interested).