Implicit Functions

Implicit Functions and Their Derivatives.

- Up till now we have only worked with functions in which the endogenous variables are explicit functions of the exogenous variables. That is, functions of the form
$$y = f(x_1, \ldots, x_n).$$
When variables are separated as above, we say that the endogeonus variable y is an *explicit function* of the exogenous variables x_i.

- Often in economic models, equations – such as those resulting from first order conditions in a maximization problem – have the exogenous variables mixed in with the endogenous, as in
$$G(x_1, \ldots, x_n, y) = 0.$$
If, for each $(x_1, \ldots, x_n)$, the above equation determines a corresponding y value, we say that the equation defines the endogenous variable y as an *implicit function* of the exogenous variables $x_1, \ldots, x_n$.

- Often it is not possible to solve such equations to get the endogenous variables on one side, and the exogenous on the other.

- We will look at a method which allows us to answer the question of how a small change in an exogenous variable changes affects the endogenous variables.

Example 1.

1. The equations
$$6x + 3y = 9, \text{ or } 6x + 3y - 9 = 0$$
express y as an implicit function of x. In this linear case, we can easily solve and write y as an explicit function of x:
$$y = 3 - 2x.$$

2. The implicit function
$$y^5 - 3xy + 2x^2 = 0$$
cannot be solved for an explicit function. However it still defines y as a function of x. For instance, when $x = 0$, we have $y^5 = 0$ with solution $y = 0$. When $x = 1$, we have $y^5 - 3y + 2 = 0$, with solution $y = 1$.

3. Consider a profit maximizing firm using a single input, labour (L), at a wage of w dollars per hour to produce a single output with production function $q = f(L)$.

   - If the price of output is p dollars, the firm's profit function for any fixed p and w is
$$\Pi(L) = pf(L) - wL.$$

- To derive the equation for the profit maximizing choice of labour, we take the derivative of the profit function with respect to L to get

$$pf'(L) - w = 0.$$

- Here we think of p and w as exogenous variables. For each choice of p and w, the firm will choose L to satisfy the above equation.
- Often we cannot solve this equation explicitly for L. However we can work with the equation as an implicit function of p and w in order to find, say, how the optimal choice of L changes as w or p increases. ⧫

- Just because we can write down an implicit function $G(x, y) = c$, it does not mean that this equation automatically defines y as a function of x.

Example 2. Consider the simple implicit function

$$x^2 + y^2 = 1. \tag{1}$$

- When $x > 1$, there is no y which satisfies (1).
- However, we usually start with a specific solution of the implicit function and ask if we vary x a little, can we still find a y near the original that satisfies the equation.
- For example if we start with the solution $x = 0$, $y = 1$ and vary x a little, we can find a unique $y = \sqrt{1 - x^2}$ near $y = 1$ that corresponds to the new x.
- However, if we start at the solution $x = 1$, $y = 0$ the no such functional relationship exists.
- If we increase x a little to $x = 1 + \varepsilon$, there is no y so that $(1 + \varepsilon, y)$ solves (1).
- It we decrease x a little to $x = 1 - \varepsilon$, there are two equally good candidates for y *near* $y = 0$, i.e.

$$y = \sqrt{2\varepsilon - \varepsilon^2} \text{ and } y = -\sqrt{2\varepsilon - \varepsilon^2}.$$

This is because the curve $x^2 + y^2 = 1$ is vertical around $(1, 0)$ – it does not define y as a function of x there. ⧫

- For a given implicit function $G(x, y) = c$ and a specified solution point (x^*, y^*), we want to know the answers to the following questions.
  1. Does $G(x, y) = c$ determine y as a continuous function of x, for x near x^* and y near y^*?
  2. If so, how do changes in x affect the corresponding $y's$?
- That is:

1. For given the implicit equation $G(x, y) = c$ and a point (x^*, y^*) such that $G(x^*, y^*) = c$ does there exist a continuous function $y = y(x)$ defined on an interval I about x^* such that
   (a) $G(x, y(x)) = c$ for all $x \in I$, and
   (b) $y(x^*) = y^*$?
2. If $y(x)$ exists and is differentiable, what is $y'(x^*)$?

Example 3. Consider the implicit function

$$x^3 - 6x^2 + 2xy + 2y^3 - 8 = 0$$

around the point $x = 2$ and $y = 2$ (which satisfies the above equation).

- Suppose that we could find a function $y = y(x)$ solving the above equation. Then
$$x^3 - 6x^2 + 2xy(x) + 2y(x)^3 - 8 = 0,$$
- Differentiating with respect to x, using the product and chain rules gives
$$3x^2 - 12x + 2y(x) + 2xy'(x) + 6y(x)^2 y'(x) = 0.$$
- Rearranging,
$$y'(x) = -\frac{3x^2 - 12x + 2y}{2x + 6y^2}.$$
- At $x = 2$, $y = 2$ we find
$$\begin{aligned} y'(x) &= -\frac{3 \cdot 2^2 - 12 \cdot 2 + 2 \cdot 2}{2 \cdot 2 + 6 \cdot 2^2} \\ &= -\frac{2}{7}. \end{aligned}$$
- Thus if there is a function $y(x)$ solving our original equation, and if it is differentiable, then as x changes by Δx, the corresponding y will change by $-2\Delta x/7$. ♦

- Consider the implicit function $G(x, y) = c$ around the point $x = x^*$, $y = y^*$.
- Suppose there is a continuously differentiable solution $y = y(x)$ to the equation $G(x, y) = c$, i.e.
$$G(x, y(x)) = c.$$
We can use the chain rule to differentiate the above with respect to x at x^*:
$$\frac{\partial G}{\partial x}(x^*, y(x^*)) + \frac{\partial G}{\partial y}(x^*, y(x^*))y'(x^*) = 0.$$

- Solving for $y'(x^*)$ gives

$$y'(x^*) = -\frac{(\partial G/\partial x)(x^*, y^*)}{(\partial G/\partial y)(x^*, y^*)}.$$

This shows that if the solution $y = y(x)$ of $G(x, y) = x$ exists and is differentiable it is necessary that $(\partial G/\partial y)(x^*, y^*)$ be nonzero.

- The following result says that this is also a sufficient condition.

Theorem 1 (Implicit Function Theorem I)**.** *Let $G(x, y)$ be a C^1 function on an open ball about (x^*, y^*) in $\mathbb{R}^2$. Suppose that $G(x^*, y^*) = c$ and consider the equation*

$$G(x, y) = c$$

If $(\partial G/\partial y)(x^*, y^*) \neq 0$, then there exists a C^1 function $y = y(x)$ defined on an open interval I about the point x^* such that

1. $G(x, y(x)) = c$ *for all* $x \in I$,
2. $y(x^*) = y^*$, *and*
3. $\dfrac{dy}{dx}(x^*) = -\dfrac{(\partial G/\partial x)(x^*, y^*)}{(\partial G/\partial y)(x^*, y^*)}$

Example 4. Let's go back to the equation $x^2 + y^2 = 1$.

- We saw this equation does determine y as a function of x about the point $x = 0$, $y = 1$.
- We can compute that $\partial G/\partial y = 2y = 2 \neq 0$ at $(0, 1)$. So the implicit function theorem tells us that $y(x)$ exists.
- It also tells us that

$$y'(0) = -\frac{\partial G/\partial x}{\partial G \partial y} = -\frac{2x}{2y} = -\frac{0}{2} = 0.$$

when $x = 0$ and $y = 1$.

- In this case we have an explicit formula for $y(x)$

$$y(x) = \sqrt{1 - x^2}.$$

- So we we can compute directly that

$$y'(x) = \frac{-x}{\sqrt{1 - x^2}},$$

which is indeeed zero when $x = 0$.

- However, no function $y(x)$ exists around $x = 1$, $y = 0$, consistent with the the implicit function theorem, since

$$\frac{\partial G}{\partial y} = 2y = 0$$

at $(1, 0)$. ♦

Theorem 2 (Implicit Function Theorem II)**.** *Let $G(x_1, \ldots, x_k, y)$ be a C^1 function on an open ball about $(x_1^* \ldots, x_k^*, y^*)$. Suppose that*

$$G(x_1^*, \ldots x_k^*, y) = c$$

If $(\partial G/\partial y)(x_1^*, \ldots, x_k^*, y^*) \neq 0$, then there exists a C^1 function $y = y(x_1, \ldots, x_k)$ defined on an open ball B about $(x_1^*, \ldots, x_k^*)$ such that

1. *$G(x_1, \ldots, x_k, y(x_1, \ldots, x_k)) = c$ for all $(x_1, \ldots, x_k) \in B$,*
2. *$y^* = y(x_1^*, \ldots, x_k^*)$, and*
3. *$\dfrac{\partial y}{\partial x_i}(x_1^*, \ldots, x_k^*) = -\dfrac{(\partial G/\partial x_i)(x_1^*, \ldots, x_k^*, y^*)}{(\partial G/\partial y)(x_1^*, \ldots, x_k^*, y^*)}$ for each i.*

Definition. A set of m equations in $m + n$ unknowns

$$\begin{aligned} G_1(x_1, \ldots, x_{m+n}) &= c_1 \\ \vdots \qquad\quad & \quad \vdots \\ G_m(x_1, \ldots, x_{m+n}) &= c_m \end{aligned} \tag{2}$$

is called a system of *implicit functions* if there is a partition of the variables into exogenous and endogenous variables, so that if one substitutes numerical values into (2) for the exogenous variables, the resulting system can be solved uniquely for corresponding values of the endogenous variables. ▲

Example 5. Consider the IS-LM model described by the linear implicit functions

$$\begin{aligned} Y &= C + I + G && \text{(national accounting identity)}, \\ C &= c_0 + c_1(Y - T) && \text{(consumption function)}, \\ I &= i_0 - i_1 r && \text{(investment function)}, \\ M_s &= m_1 Y - m_2 r && \text{(money market equilibrium)}, \end{aligned}$$

where $c_0, i_0, i_1, m_1, m_2 > 0$, and $0 < c_1 < 1$ are fixed behavioural parameters.

- We can combine these equations as

$$\begin{aligned} (1 - c_1)Y + i_1 r &= c_0 + i_0 + G - c_1 T \\ m_1 Y - m_2 r &= M_s. \end{aligned} \tag{3}$$

- The natural endogenous variables in this model are national income (Y) and the interest rate (r), the variables on the left of (3).
- The system in matrix form is

$$\begin{pmatrix} 1-c_1 & i_1 \\ m_1 & -m_2 \end{pmatrix} \begin{pmatrix} Y \\ r \end{pmatrix} = \begin{pmatrix} c_0 + i_0 + G - c_1 T \\ M_s \end{pmatrix}$$

The determinant of the above coefficient matrix is

$$d = -[(1-c_1)m_2 + i_1 m_1] < 0$$

- Therefore we can solve the system for Y and r.
- Inverting we obtain the explicit solution

$$\begin{pmatrix} Y \\ r \end{pmatrix} = \frac{1}{d} \begin{pmatrix} m_2 & i_1 \\ m_1 & 1-c_1 \end{pmatrix} \begin{pmatrix} c_0 + i_0 + G - c_1 T \\ M_s \end{pmatrix}$$

♦

- A general, linear model will have m equations in n unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{4}$$

- There will usually be a natural division of the x_i's in the model into exogenous and endogenous variables.
- This division will be successful only if, once we have chosen values for the exogenous variables, we can solve the system (4) for the endogenous variables.
- From our study of linear systems, we need exactly as many endogenous variables as equations in (4), and the square matrix corresponding to the endogenous variables must have a nonzero determinant.

Theorem 3 (Linear Implicit Function Theorem)**.** *Let $x_1, \ldots, x_k$ and $x_{k+1}, \ldots, x_n$ be a partition of the n variables of (4) into endogenous and exogenous variables, respectively. There is, for each choice of values $x^*_{k+1}, \ldots, x^*_n$ for the exogenous variables, a unique set of values $x^*_1, \ldots, x^*_k$ for the endogenous variables which solves (4) iff*

1. *$k = m$, and*
2. *the determinant of the matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{pmatrix},$$

corresponding to the endogenous variables is nonzero.

- To derive the corresponding result for nonlinear systems we
  - linearize by taking the derivative,
  - apply the linear implicit function theorem to the linearized system, and
  - transfer the results to back to the original nonlinear system.
- We write a general nonlinear system of m equations in $m+n$ unknowns as

$$\begin{aligned} G_1(y_1,\ldots,y_m,x_1,\ldots,x_n) &= c_1 \\ \vdots \qquad\qquad & \quad \vdots \\ G_m(y_1,\ldots,y_m,x_1,\ldots,x_n) &= c_m, \end{aligned} \tag{5}$$

  where we want $y_1,\ldots,y_m$ to be endogenous and $x_1,\ldots,x_n$ to be exogenous.
- We know, from the linear implicit function theorem, that we need exactly as many endogenous variables as there are independent equations, in this case m.
- Totally differentiating, the linearization of the system (5) about (y^*,x^*) is

$$\begin{aligned} \frac{\partial G_1}{\partial y_1}dy_1+\cdots+\frac{\partial G_1}{\partial y_m}dy_m+\frac{\partial G_1}{\partial x_1}dx_1+\cdots+\frac{\partial G_1}{\partial x_n}dx_n &= 0 \\ \vdots \qquad\qquad\qquad\qquad & \quad \vdots \\ \frac{\partial G_m}{\partial y_1}dy_1+\cdots+\frac{\partial G_m}{\partial y_m}dy_m+\frac{\partial G_m}{\partial x_1}dx_1+\cdots+\frac{\partial G_m}{\partial x_n}dx_n &= 0 \end{aligned} \tag{6}$$

  where all the partial derivatives are evaluated at (y^*,x^*).
- By the linear implicit function theorem, this linear system (6) can be solved for $dy_1,\ldots,dy_m$ in terms of $dx_1,\ldots,dx_n$ iff the coefficient matrix of the dy_i's

$$\begin{pmatrix} \frac{\partial G_1}{\partial y_1} & \cdots & \frac{\partial G_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial G_m}{\partial y_1} & \cdots & \frac{\partial G_m}{\partial y_m} \end{pmatrix} = \frac{\partial(G_1,\ldots,G_m)}{\partial(y_1,\ldots,y_m)}$$

  is nonsingular at (y^*,x^*).
- Since the system is linear, when the above coefficient matrix is nonsingular, we can solve the system (6) for the dy_i's in terms of the dx_j's and everything else

$$\begin{pmatrix} dy_1 \\ \vdots \\ dy_m \end{pmatrix} = -\begin{pmatrix} \frac{\partial G_1}{\partial y_1} & \cdots & \frac{\partial G_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial G_m}{\partial y_1} & \cdots & \frac{\partial G_m}{\partial y_m} \end{pmatrix}^{-1} \begin{pmatrix} \sum_{i=1}^n \frac{\partial G_1}{\partial x_i}dx_i \\ \vdots \\ \sum_{i=1}^n \frac{\partial G_m}{\partial x_i}dx_i \end{pmatrix}. \tag{7}$$

- Since the linear approximation of the original system is a true implicit function of the dy_i's in terms of the dx_j's, we conclude that the nonlinear system defines the y_i's as implicit functions of the x_j's – at least in the neighbourhood of (y^*,x^*)

- Furthermore, we can use the linear solution (7) of the dy_i's in terms of the dx_j's to find the derivatives of the y_i's with respect to the x_j's at (x^*, y^*).
- To compute $\partial y_k/\partial x_h$ we set $dx_j = 0$ for all $j \neq h$ and $dx_h = 1$ in (6) or (7) and then solve for the corresponding dy_i's. If we use (7), we get

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_h} \\ \vdots \\ \frac{\partial y_m}{\partial x_h} \end{pmatrix} = - \begin{pmatrix} \frac{\partial G_1}{\partial y_1} & \cdots & \frac{\partial G_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial G_m}{\partial y_1} & \cdots & \frac{\partial G_m}{\partial y_m} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial G_1}{\partial x_h} \\ \vdots \\ \frac{\partial G_m}{\partial x_h} \end{pmatrix}. \tag{8}$$

- Alternatively, we can use Cramer's rule on (6) to compute

$$\begin{aligned} \frac{\partial y_k}{\partial x_h} &= - \frac{\begin{vmatrix} \frac{\partial G_1}{\partial y_1} & \cdots & \frac{\partial G_1}{\partial x_h} & \cdots & \frac{\partial G_1}{\partial y_m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial G_m}{\partial y_1} & \cdots & \frac{\partial G_m}{\partial x_h} & \cdots & \frac{\partial G_m}{\partial y_m} \end{vmatrix}}{\begin{vmatrix} \frac{\partial G_1}{\partial y_1} & \cdots & \frac{\partial G_1}{\partial y_k} & \cdots & \frac{\partial G_1}{\partial y_m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \frac{\partial G_m}{\partial y_1} & \cdots & \frac{\partial G_m}{\partial y_k} & \cdots & \frac{\partial G_m}{\partial y_m} \end{vmatrix}} \\ &= - \frac{\det \frac{\partial(G_1,\ldots,G_k,\ldots,G_m)}{\partial(y_1,\ldots,x_h,\ldots,y_m)}}{\det \frac{\partial(G_1,\ldots,G_k,\ldots,G_m)}{\partial(y_1,\ldots,y_k,\ldots,y_m)}} \end{aligned} \tag{9}$$

Theorem 4 (Implicit Function Theorem III). *Let $G_1, \ldots, G_m : \mathbb{R}^{m+n} \to \mathbb{R}$ be C^1 functions. Consider the system of equations*

$$\begin{aligned} G_1(y_1, \ldots, y_m, x_1, \ldots, x_n) &= c_1 \\ \vdots \qquad\qquad & \quad \vdots \\ G_m(y_1, \ldots, y_m, x_1, \ldots, x_n) &= c_m \end{aligned} \tag{10}$$

as possibly defining $y_1, \ldots, y_m$ as implicit functions of $x_1, \ldots, x_n$. Suppose that (y^*, x^*) is a solution of (10). If the determinant of the $m \times m$ matrix

$$\begin{pmatrix} \frac{\partial G_1}{\partial y_1} & \cdots & \frac{\partial G_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial G_m}{\partial y_1} & \cdots & \frac{\partial G_m}{\partial y_m} \end{pmatrix} = \frac{\partial(G_1, \ldots, G_h, \ldots, G_m)}{\partial(y_1, \ldots, y_h, \ldots, y_m)} \tag{11}$$

evaluated at (y^*, x^*) is nonzero, then there exist C^1 functions

$$\begin{aligned} y_1 &= f_1(x_1, \ldots, x_n) \\ \vdots \quad & \quad \vdots \\ y_m &= f_m(x_1, \ldots, x_n) \end{aligned} \tag{12}$$

defined on an open ball B about x^* such that

$$\begin{aligned} G_1(f_1(x), \ldots, f_m(x), x_1, \ldots, x_n) &= c_1 \\ \vdots \qquad & \quad \vdots \\ G_m(f_1(x), \ldots, f_m(x), x_1, \ldots, x_n) &= c_m \end{aligned}$$

for all $x = (x_1, \ldots, x_m)$ in B and

$$\begin{aligned} y_1^* &= f_1(x_1^*, \ldots, x_n^*) \\ \vdots \quad & \quad \vdots \\ y_m^* &= f_m(x_1^*, \ldots, x_n^*). \end{aligned}$$

Furthermore, one can compute $(\partial f_k/\partial x_h)(y^*, x^*) = (\partial y_k/\partial x_h)(y^*, x^*)$ by setting $dx_h = 1$ and $dx_j = 0$ for $j \neq h$ in (6) and solving the resulting system for dy_k. This can be accomplished

- *by inverting the nonsingular matrix (11) to obtain the solution (8), or*
- *by applying Cramer's rule to (6) to obtain the solution (9).*

Example 6. A nonlinear generalization of the IS-LM model is the system

$$\begin{aligned} Y &= C + I + G && \text{(national accounting identity)}, \\ C &= C(Y - T) && \text{(consumption function)}, \\ I &= I(r) && \text{(investment function)}, \\ M_s &= M(Y, r) && \text{(money market equilibrium)}, \end{aligned}$$

where C, I and M satisfy

$$0 < C'(x) < 1, \; I'(r) < 0, \; \frac{\partial M}{\partial Y} > 0, \text{ and } \frac{\partial M}{\partial r} < 0. \tag{13}$$

- We can combine these equations as

$$\begin{aligned} Y - C(Y - T) - I(r) &= G \\ M(Y, r) &= M_s. \end{aligned} \tag{14}$$

- We want (14) to define Y and r as implicit functions of G, M_s, and T.
- Suppose, the current (G, M_s, T) is (G^*, M_s^*, T^*) and that the corresponding equilibrium is (Y^*, r^*).
- The linearization of the system is

$$\begin{aligned} [1 - C'(Y^* - T^*)]dY - I'(r^*)dr &= dG - C'(Y^* - T^*)dT, \\ \frac{\partial M}{\partial Y}dY + \frac{\partial M}{\partial r}dr &= dM_s, \end{aligned}$$

or

$$\begin{pmatrix} 1 - C'(Y^* - T^*) & -I'(r^*) \\ \frac{\partial M}{\partial Y} & \frac{\partial M}{\partial r} \end{pmatrix} \begin{pmatrix} dY \\ dr \end{pmatrix} = \begin{pmatrix} dG - C'(Y^* - T^*)dT \\ dM_s \end{pmatrix} \tag{15}$$

all evaluated at $(Y^*, r^*, G^*, M_s^*, T^*)$.

- The determinant of the above coefficient matrix,

$$D = [1 - C'(Y^* - T^*)]\frac{\partial M}{\partial r} + I'(r^*)\frac{\partial M}{\partial Y},$$

  is negative by (13) and so is nonzero.

- By the implicit function theorem, the system (14) does indeed define Y and r as implicit functions of G, M_s, and T around $(Y^*, r^*, G^*, M_s^*, T^*)$.

- Inverting (15), we find

$$\begin{pmatrix} dY \\ dr \end{pmatrix} = \frac{1}{D} \begin{pmatrix} \frac{\partial M}{\partial r} & I'(r^*) \\ -\frac{\partial M}{\partial Y} & 1 - C'(Y^* - T^*) \end{pmatrix} \begin{pmatrix} G - C'(Y^* - T^*)dT \\ dM_s \end{pmatrix}$$

- If we increase the money supply M_s, keeping G and T constant, we find

$$\frac{\partial Y}{\partial M_s} = \frac{1}{D} I'(r) \text{ and } \frac{\partial r}{\partial M_s} = \frac{1}{D}[1 - C'(Y^* - T^*)],$$

  so that Y rises and r falls. ♦