Set Theory Exercises

1. Which of the following are true and which are false?

   (a) $3 \in (3, 5]$.

   (b) $10 \notin (-\infty, \pi^2]$.

   (c) $7 \in \{2, 3, \dots, 11\}$.

   (d) $\pi \in (2, \infty)$.

   (e) $-1.3 \in \{\dots, -3, -2, -1\}$.

   (f) $[1, 2] \subseteq \{0, 1, 2, 3\}$.

   (g) $\{-1, 0, 1\} \subseteq [-1, 1)$.

   (h) $[5, 7] \subseteq (4, \infty)$.

   (i) $\{2, 4, 8, 16, \dots\} \subseteq [2, \infty)$.

   (j) $\{\emptyset\} \subseteq G$ for all sets G.

   (k) $\emptyset \subseteq G$ for all sets G.

   (l) $\emptyset \subseteq \mathcal{P}(G)$ for all sets G.

   (m) $\{\emptyset\} \subseteq \mathcal{P}(G)$ for all sets G.

   (n) $\emptyset \in G$ for all sets G.

   (o) $\emptyset \in \mathcal{P}(G)$ for all sets G.

   (p) $\{\{\emptyset\}\} \subseteq \mathcal{P}(\emptyset)$.

   (q) $\{\emptyset\} \subseteq \{\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}\}$.

   (r) $\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.

2. Let A, B and C be sets. Suppose $A \subseteq B$ and $B \subseteq C$ and $C \subseteq A$. Show that $A = B = C$.

3. Let $X = [0, 5)$ and $Y = [2, 4]$ and $Z = (1, 3]$ and $W = (3, 5)$ be intervals in $\mathbb{R}$. Find each of the following sets.

   (a) $Y \cup Z$.

   (b) $Z \cap W$.

   (c) $Y \setminus W$.

   (d) $X \times W$.

   (e) $(X \cap Y) \cup Z$.

   (f) $X \setminus (Z \cup W)$.

4. Let

$$\begin{aligned}
G &= \{n \in \mathbb{Z} \mid n = 2m \text{ for some } m \in \mathbb{Z}\}, \\
H &= \{n \in \mathbb{Z} \mid n = 3k \text{ for some } k \in \mathbb{Z}\}, \\
I &= \{n \in \mathbb{Z} \mid n^2 \text{ is odd}\}, \\
J &= \{n \in \mathbb{Z} \mid 0 \leq n \leq 10\}.
\end{aligned}$$

   Find each of the following sets.

(a) $G \cup I$.

(b) $G \cap I$.

(c) $G \cap H$.

(d) $J \setminus G$.

(e) $I \setminus H$.

(f) $J \cap (G \setminus H)$.

5. For real numbers a, b and c we know that $a - (b - c) = (a - b) + c$. Discover and prove a formula for $A - (B - C)$, where A, B and C are sets.

6. Let A and B be sets such that $A \neq B$. Suppose that Z is a set such that $A \times Z = B \times Z$. Prove that $Z = \emptyset$.

7. In each of the following cases, suppose we are given sets B_k for each $k \in \mathbb{N}$. Find $\bigcup_{k\in\mathbb{N}} B_k$ and $\bigcap_{k\in\mathbb{N}} B_k$.

   (a) $B_k = \{0, 1, 2, 3, \ldots, 2k\}$.

   (b) $B_k = \{k - 1, k, k + 1\}$.

   (c) $B_k = [\frac{3}{k}, \frac{5k+2}{k}) \cup \{10 + k\}$.

   (d) $B_k = [-1, 3 + \frac{1}{k}) \cup [5, \frac{5k+1}{k})$.

   (e) $B_k = (-\frac{1}{k}, 1] \cup (2, \frac{3k-1}{k}]$.

   (f) $B_k = [0, \frac{k+1}{k+2}] \cup [7, \frac{7k+1}{k})$.

8. In each of the following cases, define a family of sets $\{E_k\}_{k\in\mathbb{N}}$, where $E_k \subseteq \mathbb{R}$ for each $k \in \mathbb{N}$, where no two sets E_k are equal to each other, and such that the given conditions hold.

   (a) $\bigcup_{k\in\mathbb{N}} E_k = [0, \infty)$ and $\bigcap_{k\in\mathbb{N}} E_k = [0, 1]$.

   (b) $\bigcup_{k\in\mathbb{N}} E_k = (0, \infty)$ and $\bigcap_{k\in\mathbb{N}} E_k = \emptyset$.

   (c) $\bigcup_{k\in\mathbb{N}} E_k = \mathbb{R}$ and $\bigcap_{k\in\mathbb{N}} E_k = \{3\}$.

   (d) $\bigcup_{k\in\mathbb{N}} E_k = (2, 8)$ and $\bigcap_{k\in\mathbb{N}} E_k = [3, 6]$.

   (e) $\bigcup_{k\in\mathbb{N}} E_k = [0, \infty)$ and $\bigcap_{k\in\mathbb{N}} E_k = \{1\} \cup [2, 3)$.

   (f) $\bigcup_{k\in\mathbb{N}} E_k = \mathbb{N}$ and $\bigcap_{k\in\mathbb{N}} E_k = \{\ldots, -2, 0, 2, 4, 6, \ldots\}$.

   (g) $\bigcup_{k\in\mathbb{N}} E_k = \mathbb{R}$ and $\bigcap_{k\in\mathbb{N}} E_k = \mathbb{N}$.

9. Let I be a non-empty set, let $\{A_i\}_i \in I$ be a family of sets indexed by I and let B be a set.

   (a) Show that $B \times (\bigcup_{i \in I} A_i) = \bigcup_{i \in I}(B \times A_i)$.

   (b) Show that $B \times (\bigcap_{i \in I} A_i) = \bigcap_{i \in I}(B \times A_i)$.