Unconstrained Optimization in Euclidean Space Exercises

All the exercises are from Simon & Blume. You may find exercises marked with an asterisk harder.

1. Let $\mathcal{D} = [0, 1]$. Describe the set $f(\mathcal{D})$ in each of the following cases, and identify $\sup f(\mathcal{D})$ and $\inf f(\mathcal{D})$. In which cases does f attain its supremum? What about its infimum?

   (a) $f(x) = 1 + x$ for all $x \in \mathcal{D}$.

   (b) $f(x) = 1$ if $x < 1/2$ and $f(x) = 2x$ otherwise.

   (c) $f(x) = x$ if $x < 1$ and $f(1) = 2$.

   (d) $f(0) = 1$, $f(1) = 0$, and $f(x) = 3x$ for $x \in (0, 1)$.

2. Suppose $\mathcal{D} \subseteq \mathbb{R}^n$ is a set consisting of a finite number of points $\{x_1, \ldots, x_p\}$. Show that any function $f : \mathcal{D} \to \mathbb{R}$ has a maximum and a minimum on $\mathcal{D}$. Is this result implied by the Weierstrass theorem? Explain your answer.

3. Use the Weierstrass theorem to show that solution exists to the expenditure minimization problem in the notes, as long as the utility function u is continuous on $\mathbb{R}^n_+$ and the price vector p satisfies $p \gg 0$. What if one of these conditions fails?*

4. A consumer, who lives for two periods, has the utility function $v(c(1), c(2))$, where $c(t) \in \mathbb{R}^n_+$ denotes the consumer's consumption bundle in period t, $t = 1, 2$. The price vector in period t is given by $p(t) = (p_1(t), \ldots, p_n(t))$. The consumer has an initial wealth of W_0, but has no other income. Any amount not spent in the first period can be saved and used for the second period. Savings earn interest at a rate $r \geq 0$. (Thus a dollar saved in period 1 becomes $(1+r)$ dollars in period 2).

   (a) Set up the consumer's utility maximization problem.

(b) Show that the feasible set in this problem is compact iff $p(1) \gg 0$ and $p(2) \gg 0$.*

5. Consider the theorem in the notes which says that whenever $x \in \operatorname{int} \mathcal{D}$ is a local maximizer of f on $\mathcal{D}$ it must be that $Df(x) = 0$. Is this theorem valid if $x \notin \operatorname{int} \mathcal{D}$?

6. Exercise 16.1.

7. Rewrite the theorem on second order conditions in terms of the determinantal characteristics of positive definite and negative definite matrices.

8. Exercise 17.1-17.2.

9. Find and classify all the critical points (local maximum, local minimum, saddle point) of each of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$. Are any of the local optima also global optima?

   (a) $f(x, y) = 2x^3 + xy^2 5x^2 + y^2$.

   (b) $f(x, y) = e^{2x}(x + y^2 + 2y)$.

   (c) $f(x, y) = xy(a - x - y)$.

   (d) $f(x, y) = x \sin y$.

   (e) $f(x, y) = x^4 + x^2y^2 - y$.

   (f) $f(x, y) = x^4 + y^4 - x^3$.

   (g) $f(x, y) = \frac{x}{1+x^2+y^2}$.

   (h) $f(x, y) = (x^4/32) + x^2y^2 - x - y^2$.

10. Find the maximum and minimum values of

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the set $\{(x, y) \in \mathbb{R}^2_+ \mid x + y = 9\}$ by representing the problem as an unconstrained optimization problem in one variable.