

Relations Exercises

- For each of the following relations on \mathbb{Z} , find $[3]$, $[-3]$ and $[6]$.
 - Let R be the relation given by $a R b$ iff $a = |b|$, for all $a, b \in \mathbb{Z}$.
 - Let S be the relation given by $a S b$ iff $a|b$, for all $a, b \in \mathbb{Z}$.¹
 - Let T be the relation given by $a T b$ iff $b|a$, for all $a, b \in \mathbb{Z}$.
 - Let Q be the relation given by $a Q b$ iff $a + b = 7$, for all $a, b \in \mathbb{Z}$.
- For each of the following relations on \mathbb{R}^2 , give a geometric description of $[(0, 0)]$ and $[(3, 4)]$.
 - Let S be the relation given by $(x, y) S (z, w)$ iff $y = 3w$, for all $(x, y), (z, w) \in \mathbb{R}^2$.
 - Let T be the relation given by $(x, y) S (z, w)$ iff $x^2 + 3y^2 = 7z^2 + w^2$, for all $(x, y), (z, w) \in \mathbb{R}^2$.
 - Let Q be the relation given by $(x, y) Q (z, w)$ iff $x = z$ or $y = w$, for all $(x, y), (z, w) \in \mathbb{R}^2$.
- Is each of the following relations reflexive, symmetric and/or transitive?
 - Let S be the relation on \mathbb{R} defined by $x S y$ iff $y = |x|$, for all $x, y \in \mathbb{R}$.
 - Let P be the set of all people, and let R be the relation on P given by $x R y$ iff x and y were not born in the same city, for all people x and y .
 - Let T be the set of all triangles in the plane, and let G be the relation on T given by $s G t$ iff s has greater area than t , for all triangles $s, t \in T$.
 - Let P be the set of all people, and let M be the relation on P given by $x M y$ iff x and y have the same mother, for all people x and y .

¹We say a **divides** b if there is some $q \in \mathbb{Z}$ such that $aq = b$. If a divides b , we write $a|b$, and we say that a is a **factor** of b , and that b is **divisible** by a .

- (e) Let P be the set of all people, and let N be the relation on P given by $x N y$ iff x and y have the same colour hair or the same colour eyes, for all people x and y .
- (f) Let D be the relation on \mathbb{N} defined by $a D b$ iff $a|b$, for all $a, b \in \mathbb{N}$.
- (g) Let T be the relation on $\mathbb{Z} \times \mathbb{Z}$ defined by $(x, y) T (z, w)$ iff (x, y) and (z, w) are both on a line in \mathbb{R}^2 with slope an integer, for all $(x, y), (z, w) \in \mathbb{Z} \times \mathbb{Z}$.
4. Let A be a set, and let R be a symmetric and transitive relation on A . Find the flaw in the following alleged proof that this relation is necessarily reflexive: “Let $x \in A$. Choose $y \in A$ such that $x R y$. By symmetry we have $y R x$, and then by transitivity we have $x R x$. Hence R is reflexive.”
5. Let A be a set, and let R be a relation on A .
- (a) Suppose R is reflexive. Show that $\bigcup_{x \in A} [x] = A$.
- (b) Suppose R is symmetric. Let $x, y \in A$. Show that $x \in [y]$ iff $y \in [x]$.
- (c) Suppose R is transitive. Let $x, y \in A$. Show that if $x R y$, then $[y] \subseteq [x]$.
6. Which of the following relations is an equivalence relation?
- (a) Let M be the relation on \mathbb{R} given by $x M y$ iff $x - y$ is an integer, for all $x, y \in \mathbb{R}$.
- (b) Let S be the relation on \mathbb{R} given by $x S y$ iff $x = |y|$, for all $x, y \in \mathbb{R}$.
- (c) Let T be the relation on \mathbb{R} given by $x T y$ iff $\sin x = \sin y$, for all $x, y \in \mathbb{R}$.
- (d) Let P be the set of all people, and let Z be the relation on P given by $x Z y$ iff x and y are first cousins, for all people x and y .
- (e) Let P be the set of all people, and let R be the relation on P given by $x R y$ iff x and y have the same maternal grandmother, for all people x and y .

- (f) Let L be the set of all lines in the plane, and let W be the relation on L given by $\alpha W \beta$ iff α and β are parallel, for all $\alpha, \beta \in L$.
7. Let A be a non-empty set. Let I be a non-empty set and let $\{E_i\}_{i \in I}$ be a family of equivalence relations on A indexed by I .
- (a) Show that $\bigcap_{i \in I} E_i$ is an equivalence relation on A .
- (b) Is $\bigcup_{i \in I} E_i$ an equivalence relation on A ? Give a proof or a counterexample.
8. For each of the following partitions, describe the corresponding equivalence relation.
- (a) Let \mathcal{E} be the partition of $A = \{1, 2, 3, 4, 5\}$ given by $\mathcal{E} = \{\{1, 5\}, \{2, 3, 4\}\}$.
- (b) Let \mathcal{Z} be the partition of \mathbb{R} given by $\mathcal{Z} = \{T_x\}_{x \in \mathbb{R}}$ where $T_x = \{x, -x\}$ for all $x \in \mathbb{R}$.
- (c) Let \mathcal{D} be the partition of \mathbb{R}^2 consisting of all circles in \mathbb{R}^2 centered at the origin (the origin is considered a “degenerate” circle).
- (d) Let \mathcal{W} be the partition of \mathbb{R} given by $\mathcal{W} = \{[n, n + 2) \mid n \text{ is an even integer}\}$.
9. Is each of the following relations antisymmetric, a partial ordering and/or a total ordering?
- (a) Let A be the set of people in Australia, and let B be the relation on A given by $x B y$ iff x drinks more beer annually than y , for all $x, y \in A$.
- (b) Let W be the set of all people who ever lived and will ever live, and let C be the relation on W given by $x C y$ iff y is an ancestor of x or $y = x$, for all $x, y \in W$.
10. Let (A, \preceq) be a poset and let $X \subseteq A$
- (a) Show that if X has a least element, it is unique, and if it has a greatest element, it is unique.

- (b) Find an example of a poset that has both a minimum and a maximum, an example that has a minimum but not a maximum, an example that has a maximum but not a minimum, and an example that has neither.