Functions and Cardinality Exercises

1. Let $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$. Which of the following subsets of $A \times B$ are functions $A \to B$?

   (a) {(b,1),(c,2),(a,3)}.

   (b) {(a,3),(c,2),(a,1)}.

   (c) {(c,1),(b,1),(a,2)}.

   (d) {(a,1),(b,3)}.

   (e) {(c,1),(a,2),(b,3), (c,2)}.

   (f) {(a,3),(c,3),(b,3)}.

2. Which of the following phrases completely (and strictly) describe functions?

   (a) Let $f(x) = \cos x$.

   (b) To every person a, let $g(a)$ be a's height in inches.

   (c) For every real number, assign the real number that is the logarithm of the original number.

   (d) Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = e^x$.

3. Which of the following formulas define functions $\mathbb{R} \to \mathbb{R}$?

   (a) $f(x) = \sin x$ for all $x \in \mathbb{R}$.

   (b) $p(x) = \frac{x^2+3}{x+5}$ for all $x \in \mathbb{R}$.

   (c) $q(x) = \ln(x^4 + 1)$ for all $x \in \mathbb{R}$.

   (d) $r(x) = \begin{cases} e^x, & \text{if } x \geq 0 \\ \cos x, & \text{if } x \leq 0. \end{cases}$

   (e) $s(x) = \begin{cases} x^2, & \text{if } x \geq 1 \\ x^3, & \text{if } x \leq 0. \end{cases}$

   (f) $t(x) = \begin{cases} x^3 - 2, & \text{if } x \geq 0 \\ |x|, & \text{if } x \leq 0. \end{cases}$

   (g) $g(x) = \begin{cases} \sin x, & \text{if } x \geq \pi \\ x, & \text{if } x < \pi. \end{cases}$

4. Find the range of each of the following functions $\mathbb{R} \to \mathbb{R}$.

   (a) $f(x) = x^6 - 5$ for all $x \in \mathbb{R}$.

(b) $g(x) = x^3 - x^2$ for all $x \in \mathbb{R}$.

(c) $h(x) = xe^{x-1} + 3$ for all $x \in \mathbb{R}$.

(d) $p(x) = \sqrt{x^4 + 5}$ for all $x \in \mathbb{R}$.

(e) $q(x) = \sin x + \cos x$ for all $x \in \mathbb{R}$.

5. For each of the following functions $f : \mathbb{R} \to \mathbb{R}$ and each set $T \subset \mathbb{R}$, find $f_*(T)$, $f^*(T)$, $f_*(f^*(T))$ and $f^*(f_*(T))$.

   (a) Let f be given by $f(x) = (x+1)^2$ for all $x \in \mathbb{R}$, and let $T = [-1, 1]$.

   (b) Let f be given by $f(x) = \lfloor x \rfloor$ for all $x \in \mathbb{R}$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x (e.g. $\lfloor 3.3 \rfloor = 3$), and let $T = [0, 2] \cup (5, 7)$.

6. Let $f : A \to B$ be a function. Let $S, T \subseteq B$. Show that if $S \subseteq T$, then $f^*(S) \subseteq f^*(T)$. (Hint: begin the proof with "Suppose $S \subseteq T$. Let $x \in f^*(S) \ldots$" and then use the definition of an inverse image.)

7. For each pair of functions f and g given below, find formulas for $f \circ g$ and $g \circ f$ (simplifying when possible).

   (a) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = e^x$ for all $x \in \mathbb{R}$, and let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = \sin x$ for all $x \in \mathbb{R}$.

   (b) Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be given by $f(x) = x^7$ for all $x \in \mathbb{R}$, and let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be given by $g(x) = x^{-3}$ for all $x \in \mathbb{R}^+$.

   (c) Let $f : \mathbb{R} \to [0, \infty)$ be given by $f(x) = x^6$ for all $x \in \mathbb{R}$, and let $g : [0, \infty) \to \mathbb{R}$ be given by $g(x) = \sqrt[6]{x}$ for all $x \in [0, \infty)$.

8. For each of the following functions $f : \mathbb{R} \to \mathbb{R}$, find non-identity functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $f = h \circ g$.

   (a) $f(x) = \sqrt{x + 7}$ for all $x \in \mathbb{R}$.

   (b) $f(x) = \sqrt{x} + 7$ for all $x \in \mathbb{R}$.

   (c) $f(x) = \begin{cases} x^6, & \text{if } x \geq 0 \\ x^4, & \text{if } x < 0. \end{cases}$

(d) $f(x) = \begin{cases} x^3, & \text{if } x \geq 0 \\ x, & \text{if } x < 0. \end{cases}$

9. Let $f : A \to B$ and $g : B \to C$ be functions that both have inverse functions. Show that $g \circ f$ has an inverse function, and that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

10. Find two right inverses for each of the following functions.

    (a) Let $f : \mathbb{R} \to [0, \infty)$ be given by $f(x) = |x|$ for all $x \in \mathbb{R}$.

    (b) Let $g : \mathbb{R} \to [1, \infty)$ be given by $g(x) = e^{x^2}$ for all $x \in \mathbb{R}$.

11. Find two left inverses for each of the following functions.

    (a) Let $f : [0, \infty) \to \mathbb{R}$ be given by $f(x) = x^3 + 4$ for all $x \in [0, \infty)$.

    (b) Let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = e^x$ for all $x \in \mathbb{R}$.

12. Is each of the following functions injective, surjective, both or neither? Prove your answers.

    (a) Let $t : (1, \infty) \to \mathbb{R}$ be given by $t(x) = \ln x$ for all $x \in (1, \infty)$.

    (b) Let $s : \mathbb{R} \to \mathbb{R}$ be given by $s(x) = x^4 - 5$ for all $x \in \mathbb{R}$.

    (c) Let $g : [0, \infty) \to [0, 1)$ be given by $g(x) = \frac{x}{1+x}$ for all $x \in [0, \infty)$.

    (d) Let $k : \mathbb{R}^2 \to \mathbb{R}$ be given by $k((x, y)) = x^2 + y^2$ for all $(x, y) \in \mathbb{R}^2$.

    (e) Let $Q : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be given by $Q(n) = \{1, 2, \ldots, n\}$ for all $n \in \mathbb{N}$.

13. This exercise justifies our use of the notation $A \times B \times C$. Let A, B and C be sets. Show that there is a bijective map $g : (A \times B) \times C \to A \times (B \times C)$. (Hint: the elements of the domain are ordered pairs of the form $((a, b), c)$ while the elements of the codomain are ordered pairs of the form $(a, (b, c))$ for some $a \in A$, $b \in B$ and $c \in C$. Define a map g, using a statement of the form "let $g : (A \times B) \times C \to A \times (B \times C)$ be defined by $g((a, b), c)) = \ldots$ for all $((a, b), c) \in (A \times B) \times C$." Then show that it is injective (one-to-one) and then that it is surjective (onto) using the definitions we learnt.)

14. Show that the set of all integers that are multiples of 5 has the same cardinality as the set of all integers. (Hint: define a simple map g from the set of all integers to the set of all integers that are multiples of 5. Then show that this map is injective and surjective.)

15. Which of the following sets is countable, and which has the same cardinality as $\mathbb{R}$. Prove your claims.

    (a) $\{\sqrt[n]{2} \mid n \in \mathbb{N}\}$.

    (b) $[0,1] \times [0,1]$.

    (c) $\{9^x \mid x \in \mathbb{R}\}$.

    (Hint: Since all of the sets are infinite, to prove a set is countable you must define a map from the set to $\mathbb{N}$ – or from $\mathbb{N}$ to the set – and then show it is bijective. Similarly, to show the set has the same cardinality as $\mathbb{R}$, you need to define a map from the set to $\mathbb{R}$ – or from $\mathbb{R}$ to the set – and then show it is bijective.)

16. Show using the principle of mathematical induction that each of the following formulas holds for all $n \in \mathbb{N}$.

    (a) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

    (b) $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

    (c) $1 + 2n < 3^n$.

17. Let $a, b \in \mathbb{N}$. Show that $a^n - b^n$ is divisible by $a - b$ for all $n \in \mathbb{N}$.