Equality Constraints and the Theorem of Lagrange Exercises

All the referenece are to exercises from Simon & Blume.

1. Exercises 18.1-18.9.

2. Exercise 19.14.

3. Find the maximum and minimum of $f(x, y) = x^2 - y^2$ on the unit circle $x^2 + y^2 = 1$ using the Lagrangean method. Using the substitution $y^2 = 1 - x^2$ solve the same problem as a single variable unconstrained problem. Do you get the same results? Why or why not?

4. Show that the problem of maximizing $f(x, y) = x^3 + y^3$ on the constraint set $\mathcal{D} = \{(x, y) \mid x + y = 1\}$ has no solution. Show also that if the Lagrangean method were used on this problem, the critical points of the Lagrangean have a unique solution. Is the point identified by this solution either a local maximum or a (local or global) minimum?

5. Find the maxima and minima of the following functions subject to the specified constraints.

   (a) $f(x, y) = xy$ subject to $x^2 + y^2 = 2a^2$.

   (b) $f(x, y) = 1/x + 1/y$ subject to $(1/x)^2 + (1/y)^2 = (1/a)^2$.

   (c) $f(x, y, z) = x + y + z$ subject to $1/x + 1/y + 1/z = 1$.

   (d) $f(x, y) = x + y$ subject to $xy = 16$.

   (e) $f(x, y, z) = x^2 + 2y - z^2$ subject to $2x - y = 0$ and $x + z = 6$.

6. Consider the problem

$$\text{Minimize } x^2 + y^2 \text{ subject to } (x - 1)^3 - y^2 = 0.$$

   (a) Solve the problem geometrically.

   (b) Show that the Lagrangean method does not work in this case. Can you explain why?

7. Consider the problem of maximizing the utility function

$$u(x, y) = \sqrt{x} + \sqrt{y}$$

on the budget set $\{(x, y) \in \mathbb{R}^2_+ \mid px + y = 1\}$. Show that if the non-negativity constraints $x \geq 0$ and $y \geq 0$ are ignored, and the problem is written as an equality constrained one, the resulting Lagrangean has a unique critical point. Does this point identify a solution to the problem? Why or why not?