Differentiation Exercises

1. For each of the following functions on $\mathbb{R}$, give the set of points at which it is *not* differentiable. Diagrams will be helpful.

   (a) $e^{|x|}$

   (b) $\sin|x|$

   (c) $|\sin x|$

   (d) $|x| + |x-1|$

   (e) $|x^3 - 8|$

2. Use the *definition* of the derivative to calculate the derivatives of the following functions at the indicated points.

   (a) $f(x) = x^3$ at $x = 2$;

   (b) $g(x) = x + 2$ at $x = a$;

   (c) $h(x) = x^2 \cos x$ at $x = 0$

   (d) $p(x) = (3x+4)(2x-1)$ at $x = 1$.

   Hint: First decide what the derivative is!

3. (a) Let $h(x) = \sqrt{x} = x^{1/2}$ for $x \geq 0$. Use the definition of the derivative to prove that $h'(x) = \frac{1}{2}x^{-1/2}$ for $x > 0$.

   (b) Let $f(x) = x^{1/3}$ for $x \in \mathbb{R}$. Use the definition of the derivative to prove that $f'(x) = \frac{1}{3}x^{-2/3}$ for $x \neq 0$.

   (c) Is the function f in part (b) differentiable at $x = 0$?

4. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$.

   (a) Use the theorem in which we saw the product rule to show that f is differentiable at each $a \neq 0$ and calculate $f'(a)$. Use, without proof, the fact that $\sin x$ is differentiable and that $\cos x$ is its derivative.

   (b) Use the definition of the derivative to show that f is differentiable at $x = 0$ and that $f'(0) = 0$.

   (c) Show that f' is not continuous at $x = 0$.

5. Let $f(x) = x\sin(1/x)$ for $x \neq 0$, $f(0) = 0$.

   (a) Observe that f is continuous at $x = 0$ by 7(c) of the exercises on continuity and limits of functions.

   (b) Is f differentiable at $x = 0$? Justify your answer.

6. Let $f(x) = x^2$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$.

   (a) Sketch f.

   (b) Show that f is differentiable at $x = 0$ by using the definition.

   (c) Calculate f' on $\mathbb{R}$ and sketch.

   (d) Is f' continuous on $\mathbb{R}$? Is f' differentiable on $\mathbb{R}$?

7. Let h be given by $h(x) = (x^4 + 13x)^7$ for all $x \in \mathbb{R}$.

   (a) Calculate $h'(x)$.

   (b) Show how the chain rule justifies your calculation in (a) by writing $h = g \circ f$ for suitable f and g.

8. Repeat the last exercise for the function given by $h(x) = (\cos x + e^x)^{12}$ or any function you can think of.

9. Suppose f is differentiable at a, g is differentiable at $f(a)$ and h is differentiable at $(g \circ f)(a)$. State and prove the chain rule for $(h \circ g \circ f)'(a)$.
   Hint: Apply the chain rule for two functions twice.

10. Determine whether the conclusion of the mean value theorem holds for the following functions on the specified intervals. If the conclusion holds, give an example of a point $x \in (a, b)$ satisfying

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

   If the conclusion fails, state which hypotheses of the mean value theorem fail.

(a) $f(x) = x^2$ on $[-1, 2]$

(b) $f(x) = \sin x$ on $[0, \pi]$

(c) $f(x) = |x|$ on $[-1, 2]$

(d) $f(x) = 1/x$ on $[-1, 1]$

(e) $f(x) = 1/x$ on $[1, 3]$

(f) $f(x) = \text{sgn}(x)$ on $[-2, 2]$

11. Prove that $|\cos x - \cos y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

12. Suppose that f is differentiable on $\mathbb{R}$ and that $f(0) = 0$, $f(1) = 1$ and $f(2) = 2$.

    (a) Show that $f'(x) = 1/2$ for some $x \in (0, 2)$.

    (b) Show that $f'(x) = 1/7$ for some $x \in (0, 2)$.

13. Find the following limits if they exist

    (a) $\lim_{x\to 0}(e^{2x} - \cos x)/x$

    (b) $\lim_{x\to 0}(1 - \cos x)/x^2$

    (c) $\lim_{x\to\infty} x^3/e^{2x}$

    (d) $\lim_{x\to 0}(\sqrt{1+x} - \sqrt{1-x})/x$

    (e) $\lim_{x\to 0} x^3/(\sin x - x)$

    (f) $\lim_{x\to 0}(\tan x - x)/x^3$

    (g) $\lim_{x\to 0}[1/\sin x - 1/x]$

    (h) $\lim_{x\to 0}(\cos x)^{1/x^2}$

    (i) $\lim_{x\to\infty}(x - \sin x)/x$

    (j) $\lim_{x\to\infty} x^{\sin(1/x)}$

    (k) $\lim_{x\to 0^+}(1 + \cos x)/(e^x - 1)$

    (l) $\lim_{x\to 0}(1 - \cos 2x - 2x^2)/x^4$

    Hint: L'Hospital's rule.

14. (a) Find the Taylor series about 0 for $\cos x$ and indicate why it converges to $\cos x$ for all $x \in \mathbb{R}$.

    (b) What is the first order (linear) Taylor series approximation and what is the order of the error term?

    (c) Draw the function and its approximation.

    (d) Calculate the approximation at $x = 0.1$ and $x = 0.3$ and compare it to the function's true value at those points.

15. Repeat the previous exercise for $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$