Continuity and Limits of Functions Exercises

1. Let f be given by $f(x) = \sqrt{4-x}$ for $x \leq 4$ and let g be given by $g(x) = x^2$ for all $x \in \mathbb{R}$.

   (a) Give the domains of $f+g$, fg, $f \circ g$ and $g \circ f$.

   (b) Find the values of $(f \circ g)(0)$, $(g \circ f)(0)$, $(f \circ g)(1)$,$(g \circ f)(1)$, $(f \circ g)(2)$ and $(g \circ f)(2)$.

   (c) Are the functions $f \circ g$ and $g \circ f$ equal?

   (d) Are $(f \circ g)(3)$ and $(g \circ f)(3)$ meaningful?

2. Let f be given by $f(x) = 4$ for $x \geq 0$, $f(x) = 0$ for $x < 0$ and let g be given by $g(x) = x^2$ for all $x \in \mathbb{R}$.

   (a) Determine the following functions: $f+g$, fg, $f \circ g$ and $g \circ f$. Make sure you specify their domains.

   (b) Which of the functions f, g $f+g$, fg, $f \circ g$, $g \circ f$ is continuous?

3. The functions given by $\sin x$, $\cos x$, e^x, 2^x, $\ln x$ for $x > 0$, and x^p for $x > 0$ ($p \in \mathbb{R}$) are continuous on thier domains. Use these facts and theorems in the notes to prove that the functions given as below are also continuous.[1]

   (a) $\ln(1 + \cos^4 x)$

   (b) $[\sin^2 x + \cos^6 x]^\pi$

   (c) 2^{x^2}

   (d) 8^x

   (e) $\tan x$ for $x \neq$ odd multiple of $\pi/2$

   (f) $x^2 \sin(1/x)$ for $x \neq 0$

   (g) $x^2 \sin(1/x)$ for $x \neq 0$

   (h) $(1/x)\sin(1/x^2)$ for $x \neq 0$

[1] We have used $\cos^n x$ to denote $(\cos x)^n$ for any $n \in \mathbb{N}$ and similarly for $\sin^n x$.

4. Prove that the function $\sqrt{x}$ is continuous on its domain $[0, \infty)$. *Hint*: use the sequential definition of continuity and the fact that if (s_n) is a sequence of nonnegative real numbers and $s = \lim s_n$, then $\lim \sqrt{s_n} = \sqrt{s}$.

5. (a) Prove that if $m \in \mathbb{N}$, then the function $f(x) = x^m$ is continuous on $\mathbb{R}$. *Hint*: You can construct an ε-δ proof using the identity

$$x^m - y^m = (x - y)(x^{m-1} + x^{m-2}y + \cdots + xy^{m-2} + y^{m-1}).$$

   Or you can prove the result using induction on m. First prove that $g(x) = x$ is continuous on $\mathbb{R}$. Then use an inductive hypothesis that $f(x) = x^m$ is continuous on $\mathbb{R}$ and the theorem about the continuity of (fg).

   (b) Prove that every ***polynomial function*** $p(x) = a_0 + a_1x + \cdots + a_nx^n$ is continuous on $\mathbb{R}$.

6. A ***rational function*** is a function of the form p/q, where p and q are polyonmial functions. The domain of f is $\{x \in \mathbb{R} \mid q(x) \neq 0\}$. Prove that every rational function is continuous. *Hint*: Use the last exercise.

7. Prove that each of the following real-valued functions f is continuous at x_0 by using the ε-δ definition of continuity.

   (a) $f(x) = x^2$, $x_0 = 2$;

   (b) $f(x) = \sqrt{x}$, $x_0 = 0$;

   (c) $f(x) = x\sin(1/x)$ for $x \neq 0$ and $f(0) = 0$, $x_0 = 0$;

   (d) $f(x) = x^3$, x_0 arbitrary. *Hint*: $x^3 - x_0^3 = (x - x_0)(x^2 + x_0x + x_0^2)$.

8. Prove that the following functions are discontinuous at the indicated points x_0. Use either the sequential or the ε-δ definition.[2]

   (a) $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$, $x_0 = 0$;

[2]The function sgn is called the ***signum function***. Note that $\text{sgn}(x) = x/|x|$ for $x \neq 0$. The definition of P means P takes the value 15 on the interval $[0, 1)$, the value 28 on the interval $[1, 2)$, the value 41 on the interval $[2, 3)$, etc.

(b) $g(x) = \sin(1/x)$ for $x \neq 0$ and $g(0) = 0$, $x_0 = 0$;

(c) $\operatorname{sgn}(x) = -1$ for $x < 0$, $\operatorname{sgn}(x) = 1$ for $x > 0$ and $\operatorname{sgn}(0) = 0$, $x_0 = 0$;

(d) $P(x) = 15$ for $0 \leq x < 1$ and $P(x) = 15 + 13n$ for $n \leq x < n + 1$, x_0 a positive integer.

9. (a) Let f and g be continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove that $f(x_0) = g(x_0)$ for at least one x_0 in $[a, b]$. *Hint*: Define a function $h = f - g$ and apply the intermediate value theorem making sure to justify its use.

   (b) Show that our example about the existence of a fixed point for a continuous function from $[0, 1]$ into $[0, 1]$ can be viewed as a special case of (a).

10. Prove that $x = \cos x$ for some $x \in (0, \pi/2)$.

11. Prove that $x2^x = 1$ for some $x \in (0, 1)$.

12. Sketch the function given by $f(x) = x/|x|$ for all $x \neq 0$. Determine, by inspection, the limits $\lim_{x\to\infty} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x)$, $\lim_{x\to -\infty} f(x)$ and $\lim_{x\to 0} f(x)$ when they exist. Also indicate when they do not exist.

13. Repeat exercise 12 for $f(x) = (\sin x)/x$.

14. Repeat exercise 12 for $f(x) = x \sin(1/x)$.

15. Repeat exercise 12 for $f(x) = x^3/|x|$ and prove your assertions.

16. Find the following limits.

    (a) $\lim_{x\to a}[(x^2 - a^2)/(x - a)]$

    (b) $\lim_{x\to b}[(\sqrt{x} - \sqrt{b})/(x - b)]$, $b > 0$

    (c) $\lim_{x\to a}[(x^3 - a^3)/(x - a)]$

    Hint: Use a previous hint.

17. Prove that if $\lim_{x \to a} f(x) = 3$ and $\lim_{x \to a} g(x) = 2$, then

    (a) $\lim_{x \to a}[3f(x) + g(x)^2] = 13$

    (b) $\lim_{x \to b}[(1/g(x)] = 1/2$

    (c) $\lim_{x \to a} \sqrt{3f(x) + 8g(x)} = 5$

18. Prove that $\lim_{x \to 0^+}(1/x) = +\infty$ and $\lim_{x \to 0^-}(1/x) = -\infty$

19. Prove that $\lim_{x \to -\infty}(x - 2)^{-3} = 0$ and $\lim_{x \to 2^+}(x - 2)^{-3} = -\infty$.