Differentiation

Basic Properties of the Derivative.

Secant Tangent Applet

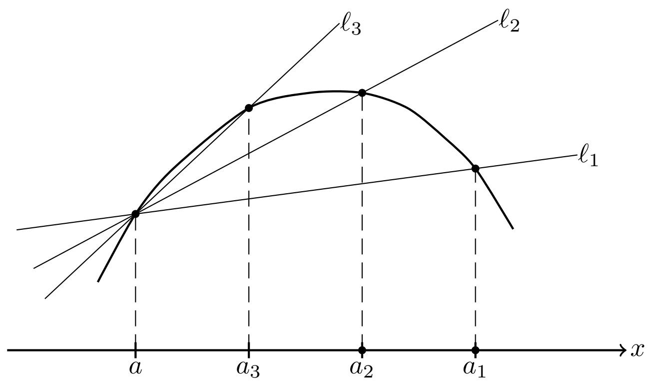

Figure 1: The derivative of f at a is the limiting value of the slope of the secants. Each of the secant lines ℓ_n through $(a, f(a))$ and $(a_n, f(a_n))$ is an approximation to the tangent line. As the points a_n get closer to a, the secant lines ℓ_n approach the tangent line of the function at a. At the same time, the slopes of the secant lines approach the slope of the tangent line to the graph of f at a. We call the limiting value of these secant slopes the derivative of f at a.

Definition. Let f be a real-valued defined on an open interval containing a point a. We say that f *is differentiable at* a, or that f *has a derivative at* a, if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite. We write $f'(a)$ for the derivative of f at a:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \qquad \blacktriangle$$

- Let $y = f(x)$ be the value of the function at x. Sometimes we write $dy/dx(a)$ or $df/dx(a)$ for the derivative of f at a.
- We will often be interested in f' as a function in its own right. The domain of f' is the set of points at which f is differentiable, and so $\operatorname{dom}(f') \subseteq \operatorname{dom}(f)$.

Example 1.

1. The derivative at a of the real-valued function g, given by $g(x) = x^2$ for all $x \in \mathbb{R}$, is calculated below.

$$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{x \to a} \frac{x^2 - a^2}{x - a} = \lim_{x \to a} (x + a) = 2a$$

   We may write $g'(x) = 2x$ since the name of the variable (a or x) does not matter. So the derivative of the function given by $g(x) = x^2$ is the function given by $g'(x) = 2x$ as you should already know.

First Derivative Applet

2. Let n be a positive integer, and let $f(x) = x^n$ for all $x \in \mathbb{R}$. We will show that $f'(x) = nx^{n-1}$ for all $x \in \mathbb{R}$. Fix an $a \in \mathbb{R}$ and notice that

$$\begin{aligned} f(x) - f(a) &= x^n - a^n \\ &= (x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}). \end{aligned}$$

Thus

$$\frac{f(x) - f(a)}{x - a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}$$

for $x \neq a$. It follows that

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= a^{n-1} + aa^{n-2} + a^2a^{n-3} + \cdots + a^{n-2}a + a^{n-1} \\ &= na^{n-1}, \end{aligned}$$

where we have used the limit theorems for functions and the fact that $\lim_{x \to a} x^k = a^k$ for $k \in \mathbb{N}$. ⧫

- The following result says that differentiability at a point implies continuity at a point.

Theorem 1. *If f is differentiable at a point a, then f is continuous at a.*

- It is important to note that the converse of the above theorem is not true. A function that is continuous at a need not be differentiable at a. In fact there are continuous functions that are not differentiable at any point in their domain.

Example 2.

1. The continuous function f given by $f(x) = |x|$ for all $x \in \mathbb{R}$ is not differentiable at $x = 0$.

2. The function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$ for all $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \notin \mathbb{Q}$ is differentiable at $x = 0$ and so continuous at $x = 0$. However, the function is discontinuous at all $x \neq 0$.

3. A famous example of a function that is continuous everywhere, but differentiable nowhere, is theWeierstrass function.

- The next theorem lists some results you may be familiar with. It allows us to compute derivatives of functions which are combinations of functions.

Theorem 2. *Let f and g be functions that are differentiable at the point a. Then*

1. $(cf)'(a) = cf'(a)$;
2. $(f+g)'(a) = f'(a) + g'(a)$;
3. $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$ *(Product Rule);*
4. $(f/g)'(a) = \dfrac{g(a)f'(a) - f(a)g'(a)}{[g(a)]^2}$, *if* $g(a) \neq 0$ *(Quotient Rule).*

Example 3. Let $h : (1, \infty) \to \mathbb{R}$ be the function given by

$$h(x) = \frac{(x^2 + 6x + 2)\sin x}{\ln x + x}$$

for all $x \in (1, \infty)$. If we let $f(x) = (x^2 + 6x + 2)\sin x$ and $g(x) = \ln x + x$, then $h = f/g$ and we can use the quotient rule (note $\ln x + x \neq 0$ for any $x \in (1, \infty)$). We have

$$\begin{aligned} h'(x) &= (f/g)'(x) \\ &= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \\ &= \frac{(\ln x + x)f'(x) - [(x^2 + 6x + 2)\sin x]g'(x)}{(\ln x + x)^2} \end{aligned}$$

To calculate the $f'(x)$ we use the product rule. Note that $f = pq$, where $p(x) = (x^2 + 6x + 2)$ and $q(x) = \sin x$, so we can use the product rule to write

$$\begin{aligned} f'(x) = (pq)'(x) &= p(x)q'(x) + p'(x)q(x) \\ &= (x^2 + 6x + 2)\cos x + (2x + 6)\sin x, \end{aligned}$$

where we used parts (1) and (2) of the last theorem to find $p'(x)$. Similarly, we can show $g'(x) = 1/x + 1$. Thus

$$\begin{aligned} h'(x) =& \frac{(\ln x + x)[(x^2 + 6x + 2)\cos x + (2x + 6)\sin x]}{(\ln x + x)^2} \\ &- \frac{[(x^2 + 6x + 2)\sin x](1/x + 1)}{(\ln x + x)^2}, \end{aligned}$$

for all $x \in (1, \infty)$ ♦

Theorem 3 (Chain Rule). *If f is differentiable at a and g is differentiable at $f(a)$, then the composite function $g \circ f$ is differentiable at a and*

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

Example 4. Let h be the function given by $h(x) = \sin(x^3 + 7x)$ for all $x \in \mathbb{R}$. We will use the chain rule to calculate the derivative of h at x. To do so, note that $h = g \circ f$ where $f(x) = x^3 + 7x$ and $g(y) = \sin y$. Then $f'(x) = 3x^2 + 7$ and $g'(y) = \cos y$, so by the chain rule

$$h'(x) = g'(f(x)) \cdot f'(x) = [\cos f(x)] \cdot f'(x) = [\cos(x^3 + 7x)] \cdot (3x^2 + 7). \quad ♦$$

Chain Rule Applet

The Mean Value Theorem.

- The next result justifies the strategy in calculus for finding the maximum and minimum of a continuous function on a closed interval $[a, b]$ (we will see later that a maximum and minimum always exist for such a problem).
- The candidates for maxima and minima are
  - the points x where $f'(x) = 0$,
  - the points where f is not differentiable, and
  - the endpoints a and b.

Theorem 4. *If f is defined on an open interval containing x_0, if f assumes its maximum or minimum at x_0, and if f is differentiable at x_0, then $f'(x_0) = 0$.*

- This theorem does *not* say that if $f'(x_0) = 0$, then f assumes its maximum or minimum at x_0. (Consider $f(x) = x^3$ on the interval $(-1, 1)$.)

Theorem 5 (Rolle's Theorem)**.** *Let f be a continuous function on $[a, b]$ that is differentiable on (a, b) and satisfies $f(a) = f(b)$. Then there exists an x in (a, b) such that $f'(x) = 0$.*

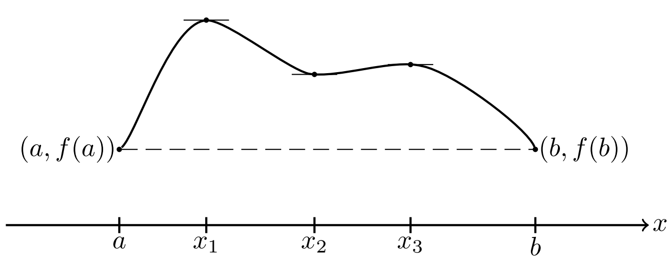

Figure 2: By Rolle's theorem, we can always find an x such that the derivative of f at x is 0. Here we can find three, x_1, x_2 and x_3.

Theorem 6 (Mean Value Theorem)**.** *Let f be a continuous function on $[a, b]$ that is differentiable on (a, b). Then there exists an x in (a, b) such that*

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

- There are several corollaries to the mean value thoerem.

Corollary 1. *Let f be a differentiable function on (a, b) such that $f'(x) = 0$ for all $x \in (a, b)$. Then f is a constant function on (a, b).*

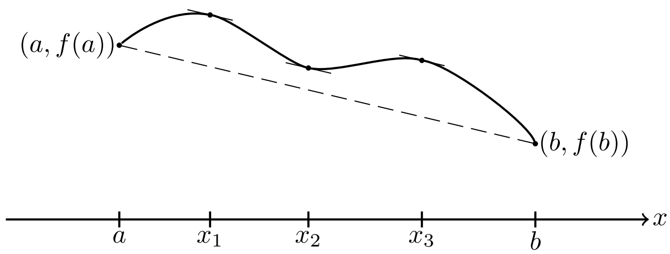

Figure 3: By the mean value theorem, we can always find an x such that the derivative of f at x is equal to the slope of the line connecting the points $(a, f(a))$ and $(b, f(b))$. In this case, we can find three such points, x_1, x_2 and x_3.

Proof. Suppose f is not constant on (a, b). Then there exist x_1, x_2 such that $a < x_1 < x_2 < b$ and $f(x_1) \neq f(x_2)$. By the mean value theorem, for some $x \in (x_1, x_2)$ we have $f'(x) = [f(x_2) - f(x_1)]/(x_2 - x_1) \neq 0$, a contradiction. ■

Corollary 2. *Let f and g be differentiable functions on (a, b) such that $f' = g'$ on (a, b). Then there exists a constant c such that $f(x) = g(x) + c$ for all $x \in (a, b)$.*

Proof. Simply apply the previous result to the function $f - g$. ■

Definition. Let f be a real-valued defined on an interval I. We say that

- f is *strictly increasing on I* if

$$x_1, x_2 \in I \quad \text{and} \quad x_1 < x_2 \quad \text{implies} \quad f(x_1) < f(x_2);$$

- f is *strictly decreasing on I* if

$$x_1, x_2 \in I \quad \text{and} \quad x_1 < x_2 \quad \text{implies} \quad f(x_1) > f(x_2);$$

- f is *increasing on I* if

$$x_1, x_2 \in I \quad \text{and} \quad x_1 < x_2 \quad \text{implies} \quad f(x_1) \leq f(x_2);$$

- f is *decreasing on I* if

$$x_1, x_2 \in I \quad \text{and} \quad x_1 < x_2 \quad \text{implies} \quad f(x_1) \geq f(x_2).$$ ▲

Corollary 3. *Let f be a differentiable function on an interval (a, b). Then*

1. *f is strictly increasing if $f'(x) > 0$ for all $x \in (a, b)$;*
2. *f is strictly increasing if $f'(x) < 0$ for all $x \in (a, b)$;*
3. *f is increasing if $f'(x) \geq 0$ for all $x \in (a, b)$;*
4. *f is decreasing if $f'(x) \leq 0$ for all $x \in (a, b)$.*

Proof of (i). Consider x_1, x_2 where $a < x_1 < x_2 < b$. By the Mean Value Theorem, for some $x \in (x_1, x_2)$ we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x) > 0.$$

Now, since $x_2 > x_1$, we see that $f(x_2) > f(x_1)$. ■

- Note that the converses of the statements in the previous corollary are not in general true.

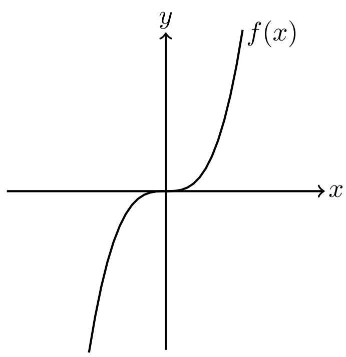

Figure 4: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ for all $x \in \mathbb{R}$ is strictly increasing on its entire domain, since $x_1 < x_2$ implies $x_1^3 < x_2^3$ for all $x_1, x_2 \in \mathbb{R}$. But $f'(0) = 3(0)^2 = 0$.

- The next theorem shows how to differentiate the inverse function of a differentiable function.

- Remember that for the inverse function to exist, the original function must be one-to-one and onto.

Theorem 7 (Inverse Function Theorem)**.** *Let f be a one-to-one continuous function on an open interval I and let $J = f(I)$. If f is differentiable at $x_0 \in I$ and if $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0) \in J$ and*

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Example 5.

1. Let f be given by $f(x) = x^3$ for all $x \in \mathbb{R}$. Then, by the inverse function theorem,
$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{3x^2},$$
for $x \neq 0$ and where $y = x^3$ i.e. $x = y^{\frac{1}{3}}$. So
$$(f^{-1})'(y) = \frac{1}{3y^{\frac{2}{3}}}.$$

We can verify this by finding the inverse function of f and then dfferentiating. Here $f^{-1}(y) = y^{\frac{1}{3}}$ for all $y \in \mathbb{R}$. Differentiation gives

$$(f^{-1})'(y) = \frac{1}{3} y^{\frac{1}{3}-1} = \frac{1}{3y^{\frac{2}{3}}}.$$

2. We can use the inverse function theorem to find the derivative of $\ln y$ if we know the derivative of e^x, since they are inverses of each other on the domain $\mathbb{R}_{++}$. Let f be given by $f(x) = e^x$ for all $x \in \mathbb{R}$. Then, by the inverse function theorem,

$$(\ln y)' = (f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{e^x},$$

where $y = e^x$. Thus the derivative of $\ln y$ is $1/y$ as expected. ♦

L'Hospital's Rule.

- In analysis we frequently encounter limits of the form

$$\lim_{x \to s} \frac{f(x)}{g(x)}$$

where s represents a, a^+, a^-, ∞ or $-\infty$.

- We saw that the limit exists and is simply

$$\frac{\lim_{x \to s} f(x)}{\lim_{x \to s} g(x)},$$

provided the limits $\lim_{x \to s} f(x)$ and $\lim_{x \to s} g(x)$ exist and are finite, and provided $\lim_{x \to s} g(x) \neq 0$.

- If these limits lead to an indeterminate form such as $0/0$ or ∞/∞, then L'Hospital's rule can often be used. Furthemore other indeterminate forms like $\infty - \infty$, 1^∞, ∞^0, 0^0, or $0 \cdot \infty$ can often be manipulated into one of these forms.

Theorem 8. *Let s signify a, a^+, a^-, ∞ or $-\infty$ where $a \in \mathbb{R}$ and suppose f and g are two differentiable functions for which the following limit exists:*

$$\lim_{x \to s} \frac{f'(x)}{g'(x)} = L.$$

If

$$\lim_{x \to s} f(x) = \lim_{x \to s} g(x) = 0$$

or if

$$\lim_{x \to s} |g(x)| = +\infty,$$

then

$$\lim_{x \to s} \frac{f(x)}{g(x)} = L.$$

Example 6.

1. Assuming the familiar properties of the trigonometric functions, it is easy to calculate $\lim_{x\to 0} \sin x/x$. Using L'Hospital's rule

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = \cos(0) = 1.$$

Note that $f(x) = \sin x$ and $g(x) = x$ satisfy the hypotheses of the theorem. Actually, since the derivative of $\sin x$ is given by

$$\lim_{x\to 0} \frac{\sin x - \sin 0}{x - 0} = \lim_{x\to 0} \frac{\sin x}{x},$$

the earlier computation is dishonest! We needed to know the limit we calculated to prove the derivative of $\sin x$ is $\cos x$.

2. We will show that

$$\lim_{x\to\infty} \frac{x^2}{e^{3x}} = 0.$$

As it is written, we have an indeterminate form ∞/∞. By L'Hospital's rule, the limit will exist as long as

$$\lim_{x\to\infty} \frac{2x}{3e^{3x}}$$

exists. Since this is another indeterminant form ∞/∞, we can use L'Hospital's rule again, to say that this limit will exist provided

$$\lim_{x\to\infty} \frac{2}{9e^{3x}}$$

exists. The last limit is 0, so we can conclude that

$$\lim_{x\to\infty} \frac{x^2}{e^{3x}} = 0.$$

3. Consider

$$\lim_{x\to 0^+} \frac{\ln x}{x}$$

if it exists. By L'Hospital's rule, this appears to be

$$\lim_{x\to 0^+} \frac{1/x}{1} = +\infty$$

but this is *incorrect*. We should *always* check the hypotheses. Here $\lim_{x\to 0^+} \ln x = -\infty$ and $\lim_{x\to 0^+} x = 0$, so neither of the theorem's hypotheses hold. In fact,

$$\lim_{x\to 0^+} \frac{\ln x}{x} = -\infty.$$

4. Consider $\lim_{x\to 0^+} x\ln x$. As written we have an indeterminate form $0\cdot(-\infty)$, since $\lim_{x\to 0^+} x = 0$ and $\lim_{x\to 0^+} \ln x = -\infty$. But we can write

$$x\ln x = \frac{\ln x}{1/x}$$

to obtain an indeterminate form $-\infty/\infty$. Then we can apply L'Hospital's rule

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = -\lim_{x\to 0^+} x = 0.$$

5. The limit $\lim_{x\to 0^+} x^x$ is of indeterminate form 0^0. We can write x^x as $e^{x\ln x}$ and note that $\lim_{x\to 0^+} x\ln x = 0$ by the last example. Since $g(x) = e^x$ is continuous at 0, we may (using our theorem on the limit of a composite function) take the function g outside the limit:

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^{\lim_{x\to 0^+} x\ln x} = e^0 = 1.$$

♦

Taylor's Theorem.

Definition. Let f be a function defined on some open interval containing $x_0 \in \mathbb{R}$. If f has derivatives of all orders at x_0, then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

is called the *Taylor series for f about x_0*. ▲

Definition. Let f be a function defined on some open interval containing $x_0 \in \mathbb{R}$. If f has derivatives of all orders up to n at x_0, then the series

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

is called the *nth order Taylor polynomial* for f about x_0. ▲

Definition. The *remainder* R_n of an $(n-1)$th order Taylor polynomial is defined by

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$ ▲

- The remainder is important because, for any x,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \quad \text{iff} \quad \lim_{n\to\infty} R_n(x) = 0.$$

- It may be the case that $\lim_{n\to\infty} R_n(x) = 0$ does not hold and f is not given by its Taylor series.
- The following theorems give us a condition under which f is given by its Taylor series.
- The first theorem tells us something about the value of this remainder.

Theorem 9 (Taylor's Theorem)**.** *Let f be defined on (a, b), and suppose the nth derivative $f^{(n)}$ exists on (a, b). Then for any x and x_0 in (a, b) we have*

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n)}(y)}{n!}(x - x_0)^n$$

for some y between x and x_0.

- This theorem says that the remainder is

  $$R_n(x) = \frac{f^{(n)}(y)}{n!}(x - x_0)^n$$

  for some y betwen x and x_0.
- Before we state the corollary giving our condition under which a function is equal to its Taylor series, we present a definition.

Definition. A real-valued function f is said to be *bounded* if $\{f(x)|x \in \operatorname{dom}(f)\}$ is a bounded set, i.e. if there exists a real number M such that $|f(x)| \leq M$ for all $x \in \operatorname{dom}(f)$. ▲

Corollary 4. *Let f be defined on (a, b), where $x_0 \in (a, b)$ and suppose all the derivatives $f^{(n)}$ exist on (a, b) and are bounded by a single constant C. Then*

$$\lim_{n\to\infty} R_n(x) = 0 \quad \textit{for all} \quad x \in (a, b).$$

Example 7. Let $f(x) = e^x$ for all $x \in \mathbb{R}$. Then $f^{(n)} = e^x$ for all $n = 0, 1, 2, \ldots$ and so $f^{(0)} = 1$ for all n. The Taylor series for e^x about 0 is

$$\sum_{k=0}^{\infty} \frac{1}{k!}x^k.$$

For any bounded interval $(-M, M)$ in $\mathbb{R}$ all the derivatives of f are bounded (by e^M), and so by the corollary

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots .$$

Taylor Polynomial Applet

If we truncate the series to the $k = 1$ term we get

$$e^x \approx 1 + x$$

which is the first order or linear approximation of e^x. The error term is $R_2 = e^x - (1 + x)$. By Taylor's theorem we can also write this as $R_2 = e^y x^2/2$ for some y between 0 and x. The error term is second order. Similarly if we truncate the Taylor series to the $k = 2$ term, we get the quadratic approximation about 0:

$$e^x \approx 1 + x + \frac{1}{2}x^2,$$

The error term is $R_3 = e^x - (1 + x + 1/2x^2)$ or $R_3 = e^z x^3/6$ for some z between 0 and x and is third order. By the corollary as the order of the approximation approaches $+\infty$, the error approaches 0. ♦