Continuity and Limits of Functions

Continuous Functions.

- A continuous function is one we can draw without taking our pen off the paper

Definition. Let f be a real-valued function whose domain is a subset of $\mathbb{R}$.

- The function f is *continuous at* x_0 in $\mathrm{dom}(f)$ if, for every sequence (x_n) in $\mathrm{dom}(f)$ converging to x_0, we have $\lim f(x_n) = f(x_0)$.
- If f is continuous at each point of a set $S \subseteq \mathrm{dom}(f)$, then f is said to be *continuous on* S.
- The function f is said to be *continuous* if it is continuous on $\mathrm{dom}(f)$. ▲

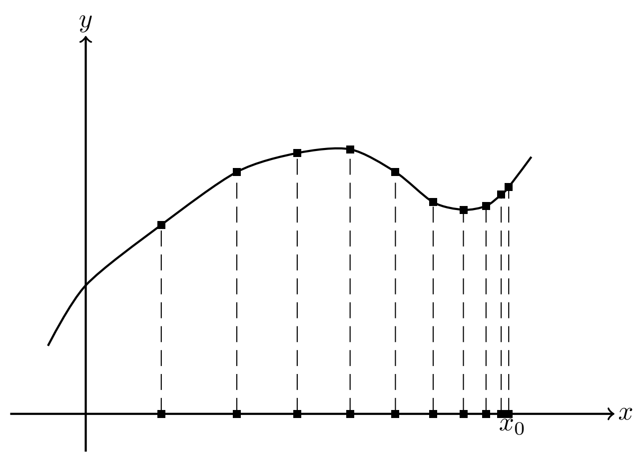

Figure 1: A continuous function. For any sequence of points $(x_n)_{n\in\mathbb{N}}$ converging to x_0, the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x_0)$.

Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x^2 + 1$ for all $x \in \mathbb{R}$. Prove f is continuous on $\mathbb{R}$.

Proof. Suppose we have a real-valued sequence (x_n) converging to x_0 i.e. $\lim x_n = x_0$. Then we have

$$\lim f(x_n) = \lim[2x_n^2 + 1] = 2[\lim x_n^2] + 1 = 2x_0^2 + 1 = f(x_0),$$

where the second equality follows by application of the limit theorems. We have shown that for any sequence (x_n) converging to x_0, we have $\lim f(x_n) = f(x_0)$.This proves that f is continuous at each $x_0 \in \mathbb{R}$ and so f is continuous on $\mathbb{R}$. ■

Example 2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = (1/x)\sin(1/x^2)$ for $x \neq 0$ and $f(0) = 0$. Show that f is discontinuous at 0.

Epsilon Delta Applet

Proof. It is sufficient to find a sequence (x_n) converging to 0 such that $(f(x_n))$ does not converge to $f(0) = 0$. To find such a sequence, we will rearrange $(1/x_n)\sin(1/x_n^2) = 1/x_n$ where $x_n \to 0$. Thus, we want $\sin(1/x_n^2) = 1$. For this we need $1/x_n^2 = 2n\pi + \pi/2$, and it follows that

$$x_n = \frac{1}{\sqrt{2n\pi + \frac{\pi}{2}}}.$$

Then $\lim x_n = 0$, while $\lim f(x_n) = \lim(1/x_n) = +\infty \neq 0$. ■

- The sequential definition of continuity implies that the values $f(x)$ are close to $f(x_0)$ when the values x are close to x_0. The following theorem provides an alternative definition of continuity.

Theorem 1 (ε-δ definition of continuity)**.** *Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Then f is continuous at $x_0 \in \mathrm{dom}(f)$ iff*

for each $\varepsilon > 0$ there exists $\delta > 0$ such that
$x \in \mathrm{dom}(f)$ and $|x - x_0| < \delta$ imply $|f(x) - f(x_0)| < \varepsilon$.

- If we draw two horizontal lines, no matter how close together, we can always cut off a vertical strip of the plane by two vertical lines in such a way that all that part of the curve which is contained in the strip lies between the two horizontal lines.

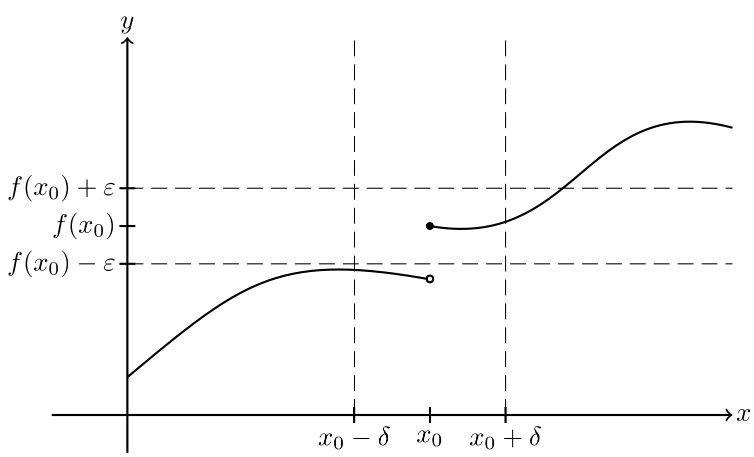

Figure 2: ε-δ definition of continuity

Example 3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 \sin(1/x)$ for all $x \neq 0$ and $f(0) = 0$. Prove f is continuous at 0.

Proof. Let $\varepsilon > 0$. Clearly $|f(x) - f(0)| = |f(x)| \leq x^2$ for all x. We want this to be less than ε, so set $\delta = \sqrt{\varepsilon}$. Then $|x - 0| < \delta$ implies $x^2 < \delta^2 = \varepsilon$, and so

$$|x - 0| < \delta \text{ implies } |f(x) - f(0)| < \varepsilon.$$

So f is continuous at 0. ■

Example 4. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x^2 + 1$ for all $x \in \mathbb{R}$. Then f is continuous on $\mathbb{R}$.

Proof. We now use the ε-δ property. Let $x_0 \in \mathbb{R}$, and let $\varepsilon > 0$. We want to show that $|f(x) - f(x_0)| < \varepsilon$ provided $|x - x_0|$ is sufficiently small , i.e. less than some δ. First note that

$$\begin{aligned} |f(x) - f(x_0)| &= |2x^2 + 1 - (2x_0^2 + 1)| = |2x^2 - 2x_0^2| \\ &= |2(x - x_0)(x + x_0)| = 2|x - x_0||x + x_0|. \end{aligned}$$

So, what we need is to find a bound for $|x + x_0|$ that does not depend on x.

Now, if $|x - x_0| < 1$ then $|x| < |x_0| + 1$ and hence $|x + x_0| \leq |x| + |x_0| < 2|x_0| + 1$. So, we have

$$|f(x) - f(x_0)| < 2|x - x_0|(2|x_0 + 1)$$

if $|x - x_0| < 1$. To arrange for $2|x - x_0|(2|x_0 + 1) < \varepsilon$ it is sufficient to have $|x - x_0| < \varepsilon/[2(2|x_0 + 1)]$ and also $|x - x_0| < 1$. So let

$$\delta = \min\left\{1, \frac{\varepsilon}{2(2|x_0| + 1)}\right\}$$

The working above shows that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. ■

- We can form new functions from old functions in several ways.

Definition. Let $A \subseteq \mathbb{R}$ and let $B \subseteq \mathbb{R}$. Consider two functions $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ and let $k \in \mathbb{R}$. We define the functions into $\mathbb{R}$ as follows.

- $|f|$ given by $|f|(x) = |f(x)|$ for all $x \in A$;
- kf given by $(kf)(x) = kf(x)$ for all $x \in A$;
- $f + g$ given by $(f + g)(x) = f(x) + g(x)$ for all $x \in A \cap B$;
- fg given by $(fg)(x) = f(x)g(x)$ for all $x \in A \cap B$;
- f/g given by $(f/g)(x) = f(x)/g(x)$ for all $x \in A \cap B$ such that $g(x) \neq 0$. ▲

Theorem 2. *Let f and g be real-valued functions that are continuous at x_0 in $\mathbb{R}$ and let $k \in \mathbb{R}$. Then*

1. *$|f|$ is continuous at x_0;*
2. *kf is continuous at x_0;*

3. *$f + g$ is continuous at x_0;*

4. *fg is continuous at x_0;*

5. *f/g is continuous at x_0 if $g(x_0) \neq 0$.*

Theorem 3. *If f is continuous at x_0 and g is continuous at $f(x_0)$, then the composite function $g \circ f$ is continuous at x_0.*

Properties of Continuous Functions.

Theorem 4 (Intermediate Value Theorem)**.** *If f is a continuous real-valued function on an interval I, then f has the intermediate value property on I: Whenever $a, b \in I$, $a < b$ and y lies between $f(a)$ and $f(b)$, there exists at least one $x \in (a, b)$ such that $f(x) = y$.*

- This theorem can be used to establish that a continuous function f has a *fixed point*, i.e. a point $x_0 \in \mathrm{dom}(f)$ such that $f(x_0) = x_0$.

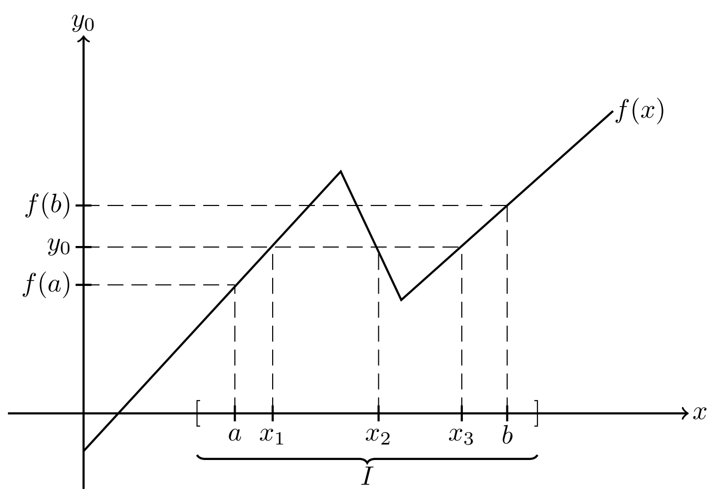

Figure 3: By the intermediate value theorem, for any $f(a) < y < f(b)$, we can find an x such that $f(x) = y$. In the case of y_0, we can find three i.e. $y_0 = f(x_1) = f(x_2) = f(x_3)$.

Example 5. Show that the continuous function $f : [0, 1] \to [0, 1]$ has a fixed point $x_0 \in [0, 1]$.

- We will use the IVT. We will define a new function g on I, and consider $a = 0$ and $b = 1$.

  Solution. Consider the function $g(x) = f(x) - x$, which is also continuous on $[0, 1]$ by (3). Now
  $$g(0) = f(0) - 0 = f(0) \geq 0$$

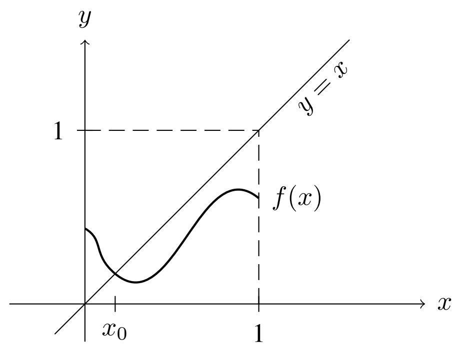

Figure 4: Fixed point. Kinda obvious?

and

$$g(1) = f(1) - 1 \leq 1 - 1 = 0.$$

So, by the intermediate value theorem, we have that $g(x_0) = 0$ for some $x_0 \in [0, 1]$. Then clearly $f(x_0) = x_0$. ■

Limits of Functions.

Definition. Let S be a subset of $\mathbb{R}$, let a be a real number or symbol ∞ or $-\infty$ that is the limit of some sequence in S, and let L be a real number or symbol ∞ or $-\infty$. We write $\lim_{x \to a^S} f(x) = L$ if

- f is a function defined on S, and
- for every sequence (x_n) in S with limit a, we have $\lim_{n\to\infty} f(x_n) = L$. ▲
- The expression "$\lim_{x\to a^S} f(x)$" is read "limit, as x tends to a along S, of $f(x)$".
- Using this definition, we see that a function f is continuous at a in $\operatorname{dom}(f) = S$ iff $\lim_{x\to a^S} f(x) = f(a)$.

Definition. For a function f and $a \in \mathbb{R}$ we write

1. $\lim_{x\to a} f(x) = L$ provided $\lim_{x\to a^S} f(x) = L$ for some set $S = J \backslash a$ where J is an open interval containing a. $\lim_{x\to a} f(x)$ is called the *(two-sided) limit of f at a.*
2. $\lim_{x\to a^+} f(x) = L$ provided $\lim_{x\to a^S} f(x) = L$ for some open interval $S = (a, b)$. $\lim_{x\to a^+} f(x)$ is called the *right-hand limit of f at a.*
3. $\lim_{x\to a^-} f(x) = L$ provided $\lim_{x\to a^S} f(x) = L$ for some open interval $S = (c, a)$. $\lim_{x\to a^-} f(x)$ is called the *left-hand limit of f at a.*
4. $\lim_{x\to\infty} f(x) = L$ provided $\lim_{x\to\infty^S} f(x) = L$ for some interval $S = (c, \infty)$.
5. $\lim_{x\to-\infty} f(x) = L$ provided $\lim_{x\to-\infty^S} f(x) = L$ for some interval $S = (-\infty, b)$. ▲

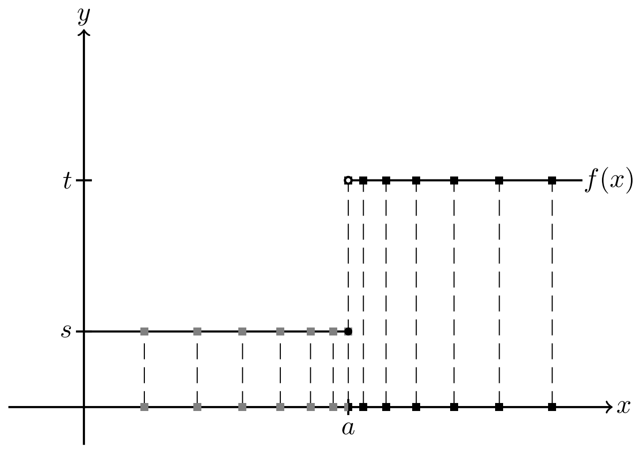

Figure 5: The right hand limit at a is t, while the left hand limit is s.

Example 6.

1. We have $\lim_{x\to 4} x^3 = 64$ and $\lim_{x\to 2}(1/x) = 1/2$ because the functions x^3 and $1/x$ are continuous at 4 and 2 respectively. One can easily show that $\lim_{x\to 0^+}(1/x) = +\infty$ and $\lim_{x\to 0^-}(1/x) = -\infty$. It follows that $\lim_{x\to 0}(1/x)$ does not exist (see theorem 7).

2. Consider
$$\lim_{x\to 2}\left[\frac{x^2-4}{x-2}\right].$$
The function we are finding the limit of is not defined at $x = 2$. We can rewrite the function as
$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2 \text{ for } x \neq 2.$$

1. Now we can see that
$$\lim_{x\to 2}\left[\frac{x^2-4}{x-2}\right] = \lim_{x\to 2}(x+2) = 4.$$
It is important to note that the functions given by $(x^2-4)/(x-2)$ and $(x+2)$ are not identical. The domain of the first is $(-\infty, 2)\cup(2, +\infty)$, while the domain of the second is $\mathbb{R}$.

3. Consider
$$\lim_{x\to 1}\left[\frac{\sqrt{x}-1}{x-1}\right].$$
To find the limit, we multiply the numerator and denominator of the function by $\sqrt{x}+1$ to get
$$\frac{\sqrt{x}-1}{x-1} = \frac{x-1}{(x-1)(\sqrt{x}+1)} = \frac{1}{\sqrt{x}+1} \text{ for } x \neq 1.$$

1. Now we can see that

$$\lim_{x\to 1}\left[\frac{\sqrt{x}-1}{x-1}\right]=\lim_{x\to 1}\left[\frac{1}{\sqrt{x}+1}\right]=\frac{1}{2}.$$

This in fact shows that if $h(x)=\sqrt{x}$ then $h'(1)=1/2$ (as you will see when we look at differentiation).

4. Let f be a real-valued function given by $f(x)=1/(x-2)^3$ for all $x\neq 2$. Then

- $\lim_{x\to+\infty} f(x)=0$,
- $\lim_{x\to-\infty} f(x)=0$,
- $\lim_{x\to 2^+} f(x)=+\infty$,
- $\lim_{x\to 2^-} f(x)=-\infty$.

Proof. We will show that $\lim_{x\to\infty} f(x)=0$. By definition, it is enough to show that $\lim_{x\to\infty^S} f(x)=0$ for $S=(2,+\infty)$. So consider any sequence (x_n) with $x_n\in S$ for all $n\in\mathbb{N}$, such that $\lim_{n\to+\infty} x_n=+\infty$ and show that

$$\lim_{n\to+\infty} f(x_n)=\lim_{n\to+\infty}\left[\frac{1}{(x_n-2)^3}\right]=0.$$

To prove the above assertion we could use our limit theorems. Instead we will prove it directly.
Let $\varepsilon>0$. We need to find an N such that for $n>N$, we have $|(x_n-2)^{-3}|<\varepsilon$. This inequality can be rearranged as $\varepsilon^{-1}<|(x_n-2)^3|$ or $\varepsilon^{-\frac{1}{3}}<|x_n-2|$. We need $x_n>\varepsilon^{-\frac{1}{3}}+2$, for the previous inequality to be satsified.

Now we use the fact that $\lim_{n\to+\infty} x_n=+\infty$. By definition of an infinite limit of a sequence, for any $M>0$ we can find an N such that $n>N$ implies $x_n>M$. Thus, if we set $M=\varepsilon^{-\frac{1}{3}}+2$, there exists an N such that

$$n>N \text{ implies } x_n>\varepsilon^{-\frac{1}{3}}+2$$

Then reversing the steps above, we find that

$$n>N \text{ implies } |(x_n-2)^{-3}|<\varepsilon.$$

This shows that $\lim_{n\to+\infty} f(x_n)=0$ for any sequence (x_n) in S such that $\lim_{n\to+\infty} x_n=+\infty$ and the result follows by definition. ■

- The following result allows us to avoid sequences and provide ε-δ definitions of a function's limits.

Theorem 5. *Let f be a function defined on a subset S of $\mathbb{R}$, let a be a real number that is the limit of some sequence in S, and let L be a real number. Then $\lim_{x\to a^S} f(x)=L$ iff*

for each $\varepsilon>0$ there exists $\delta>0$ such that
$x\in S$, $|x-a|<\delta$ imply $|f(x)-L|<\varepsilon$.

- This theorem has a number of corollaries, which are listed in the next theorem. These give us alternative definitions for the limit of a function, its lateral limits, and its limits at infinity.

Theorem 6. *1. Let f be a function defined on $J\backslash\{a\}$ for some open interval J containing a, and let L be a real number. Then $\lim_{x\to a} f(x) = L$ iff*

> *for each $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x) - L| < \varepsilon$.*

2. Let f be a function defined on some interval (a, b), and let L be a real number. Then $\lim_{x\to a^+} f(x) = L$ iff

> *for each $\varepsilon > 0$ there exists $\delta > 0$ such that $a < x < a + \delta$ implies $|f(x) - L| < \varepsilon$.*

3. Let f be a function defined on some interval (c, a), and let L be a real number. Then $\lim_{x\to a^-} f(x) = L$ iff

> *for each $\varepsilon > 0$ there exists $\delta > 0$ such that $a - \delta < x < a$ implies $|f(x) - L| < \varepsilon$.*

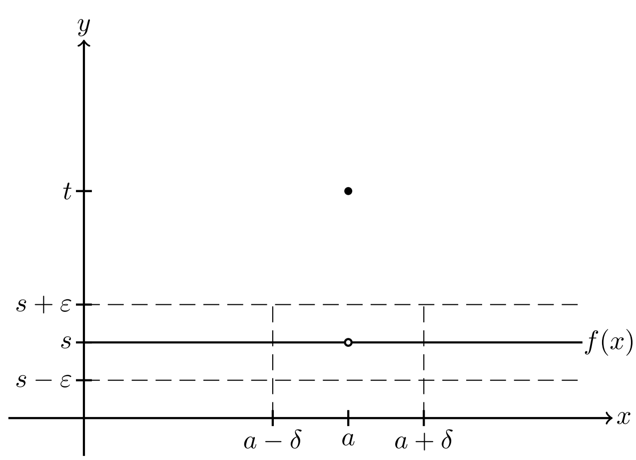

Figure 6: The (two-sided) limit at a is s, while $f(a) = t$. (This function is not continuous at a.)

4. Let f be a function defined on some interval $(c, +\infty)$, and let L be a real number. Then $\lim_{x\to+\infty} f(x) = L$ iff

> *for each $\varepsilon > 0$ there exists $\delta < +\infty$ such that $x > \delta$ implies $|f(x) - L| < \varepsilon$.*

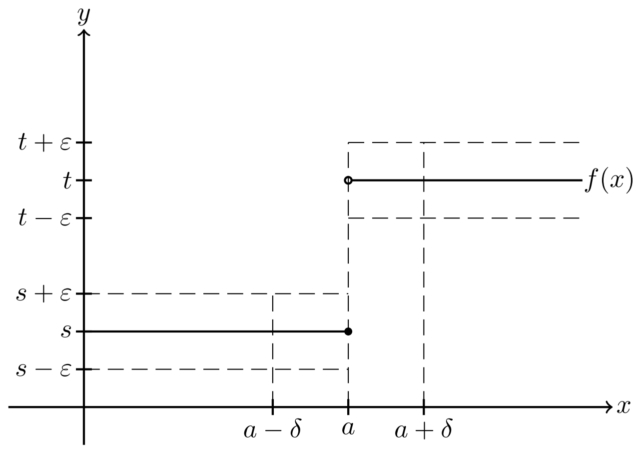

Figure 7: The right hand limit at a is t, while the left hand limit is s.

5. *Let f be a function defined on some interval $(-\infty, b)$, and let L be a real number. Then $\lim_{x\to-\infty} f(x) = L$ iff*

   for each $\varepsilon > 0$ there exists $\delta > -\infty$ such that $x < \delta$ implies $|f(x) - L| < \varepsilon$.

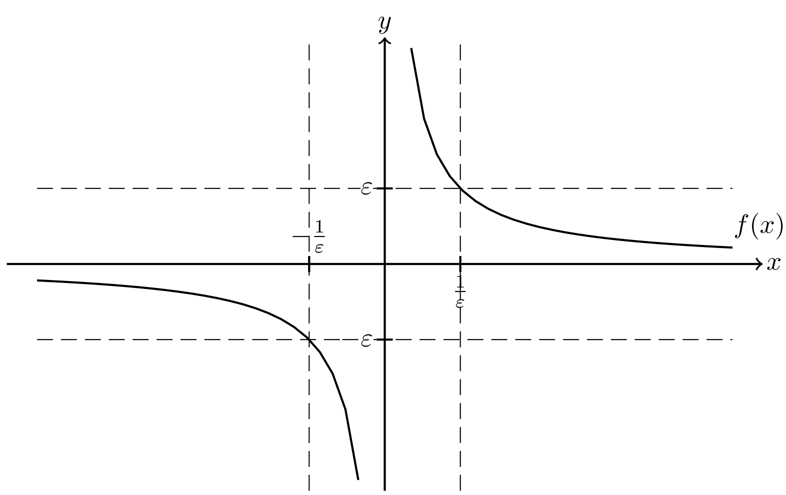

Figure 8: The function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is given by $f(x) = 1/x$ for all $x \in \mathbb{R} \setminus \{0\}$. The limits at $+\infty$ and $-\infty$ are both 0.

Example 7. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be given by $f(x) = 1/x$ for all $x \in \mathbb{R} \setminus \{0\}$ (this is the function drawn in the previous figure). Then $\lim_{x\to+\infty} = \lim_{x\to-\infty} = 0$.

Proof. We will prove the first limit. First note that the function is defined on some interval $(c, +\infty)$ – we could take any $c > 0$. Also the limit $L = 0$ is a real number.

So we use definition 4. Let $\varepsilon > 0$. We need to find a finite δ such that $x > \delta$ implies $|1/x| < \varepsilon$. If we take $\delta = 1/\varepsilon$, then the desired condition is satisfied. ■

6. *Let f be a function defined on $J \setminus \{a\}$. Then $\lim_{x\to a} f(x) = \infty$ iff*

   for each $M > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $f(x) > M$.

7. *Let f be a function defined on $J \setminus \{a\}$. Then $\lim_{x\to a} f(x) = -\infty$ iff*

   for each $M < 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $f(x) < M$.

- We can also mix the previous limit concepts, and define the following limits.

  - $\lim_{x\to a^+} f(x) = +\infty$,
  - $\lim_{x\to a^+} f(x) = -\infty$,
  - $\lim_{x\to a^-} f(x) = +\infty$,
  - $\lim_{x\to a^-} f(x) = -\infty$,
  - $\lim_{x\to +\infty} f(x) = +\infty$,
  - $\lim_{x\to +\infty} f(x) = -\infty$,
  - $\lim_{x\to -\infty} f(x) = +\infty$,
  - $\lim_{x\to -\infty} f(x) = +\infty$.

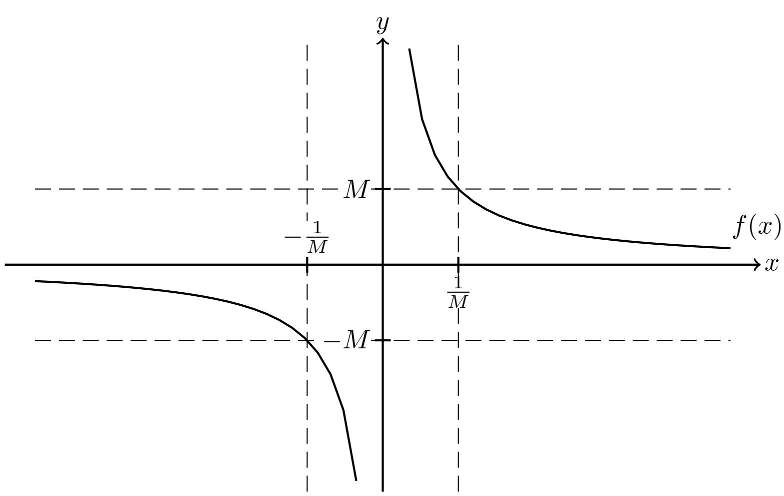

Figure 9: The function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is given by $f(x) = 1/x$ for all $x \in \mathbb{R} \setminus \{0\}$. Here $\lim_{x\to 0^+} f(x) = +\infty$, while $\lim_{x\to 0^-} f(x) = -\infty$. The two-sided limit at a does not exist. It cannot be $+\infty$ because there is no δ such that $x \in (-\delta, +\delta)$ implies $f(x) > M$.

- The next result tells us that if the left-hand and right-hand limits at a of a function exist *and are equal*, then the two-sided limit at a exists and is equal to these two limits.

- It also says the converse is also true – if the limit exists, then both the left-hand and right-hand limits exist and are equal to the two-sided limit.

Theorem 7. *Let f be a function defined on $J \setminus \{a\}$ for some open interval J containing a. Then $\lim_{x \to a} f(x)$ exists iff the limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ both exist and are equal, in which case all three limits are equal.*

Theorem 8. *Let f_1 and f_2 be functions for which the limits $L_1 = \lim_{x \to a^S} f_1(x)$ and $L_2 = \lim_{x \to a^S} f_2(x)$ exist and are finite. Then*

1. $\lim_{x \to a^S} (f_1 + f_2)(x) = L_1 + L_2$;
2. $\lim_{x \to a^S} (f_1 f_2)(x) = L_1 L_2$;
3. $\lim_{x \to a^S} (f_1/f_2)(x) = L_1/L_2$ *if* $L_2 \neq 0$ *and* $f_2(x) \neq 0$ *for* $x \in S$.

Proof of (iii). blah ■

- The following result provides conditions under which we can change the order of taking the limit and applying the function g.

Theorem 9. *Let f be a function for which the limit $L = \lim_{x \to a^S} f(x)$ exists and is finite. If g is a function defined on $\{f(x) \mid x \in S\} \cup \{L\}$ that is continuous at L, then $\lim_{x \to a^S} g \circ f(x)$ exists and equals $g(L)$.*

Example 8. Compute $\lim_{x \to 1} \sqrt{x^3 + 1}$.

Proof. Here the function $g : \mathbb{R}_+ \to \mathbb{R}$, given by $g(x) = \sqrt{x}$ for all $x \in \mathbb{R}_+$ is continuous. Thus

$$\lim_{x \to 1} \sqrt{x^3 + 1} = \sqrt{\lim_{x \to 1} x^3 + 1} = \sqrt{1 + 1} = \sqrt{2}.$$ ■